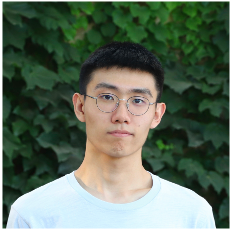
A Simple Deterministic Approximation Algorithm for Total Variation Distance between Product Distributions

Tianren Liu (Peking University)



Joint work with

Weiming Feng
University of Edinburgh

Liqiang Liu
Peking University

Problem: Given two distributions P, Q

Compute the **total variation distance**

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The **total variation distance** ("the" statistical distance)

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. . . .

Assume cryptography exists,

Hard to tell if $\Delta_{\mathrm{TV}}(P, Q) = 0$ and $\Delta_{\mathrm{TV}}(P, Q) = 1$.

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$$P(x_1, \ldots, x_n) = P_1(x_1) \cdot P_2(x_2) \cdot \cdots \cdot P_n(x_n)$$

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$\#P$-hard to compute **exactly**, even on boolean domain
[Bhattacharyya-Gayen-Meel-Myrisiotis-Pavan-Vinodchandran'22]

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output an estimation with up to ε **relative error**

$$(1-\varepsilon)\,\Delta_{\mathsf{TV}}(P,Q) \le \text{estimation} \le \Delta_{\mathsf{TV}}(P,Q)$$

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Feng-Guo-Jerrum-Wang'23

Randomized algorithm, in time $O(qn^2\varepsilon^{-2}\log\frac{1}{\delta})$

- q: domain size
- ε: relative error
- δ: error probability

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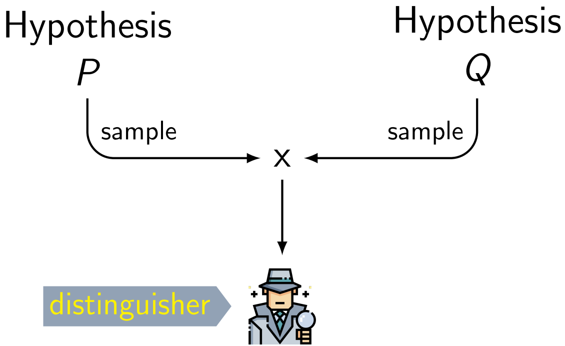
This work: Feng-Liu-Liu

Deterministic algorithm, in time $O(qn^2\varepsilon^{-1}\log\frac{n}{\varepsilon\,\Delta_{\mathrm{TV}}(P,Q)})$

Decision Theory

A decision problem: P v.s. Q

Decision Theory



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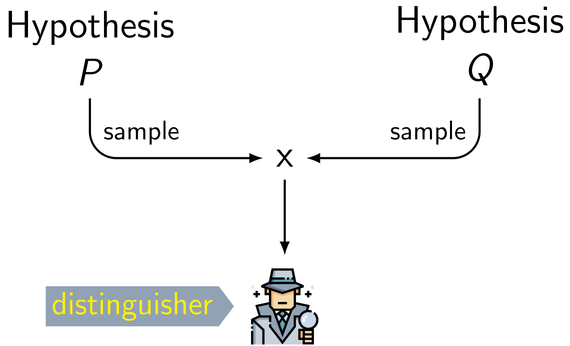
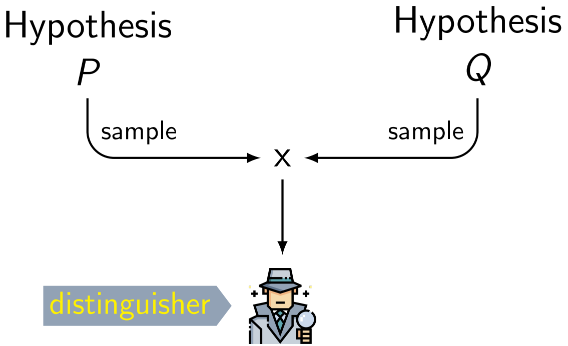


$\Delta_{\mathrm{TV}}(P, Q) = \max$  $\Big(\Pr_{x \leftarrow P}[$  $(x) \to 1] - \Pr_{x \leftarrow Q}[$  $(x) \to 1] \Big)$

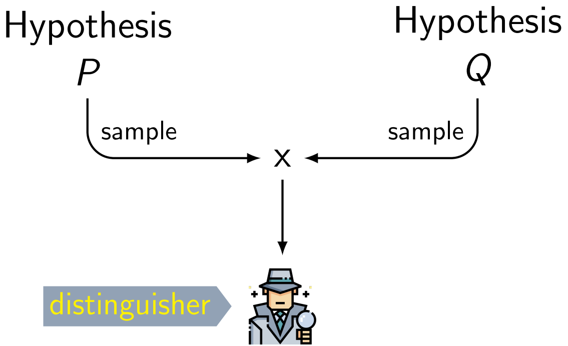
Decision Theory



The best distinguisher: **Likelihood-Ratio Test**

 checks if $P(x) \overset{?}{>} Q(x)$.

Decision Theory



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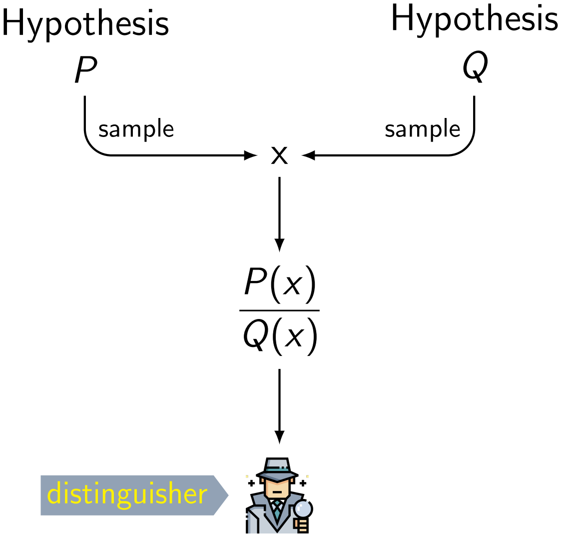

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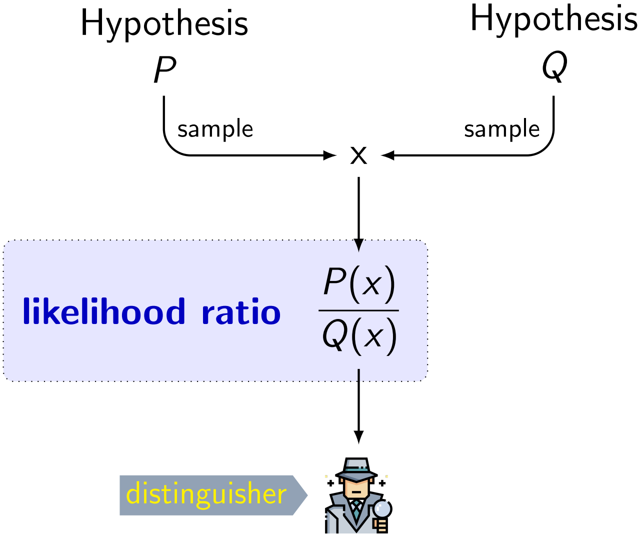

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Ratio, denoted by $R \equiv (P \| Q)$, is a random variable

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Example I

Peking		Qinghua	
♀	♂	♀	♂
$1/2$	$1/2$	$1/4$	$3/4$

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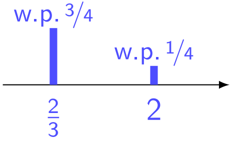

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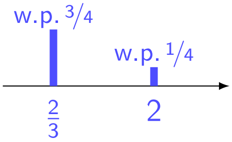

Example II

Peking				Qinghua			
♀L	♀S	♂L	♂S	♀L	♀S	♂L	♂S
1/4	1/4	2%	48%	1/8	1/8	3%	72%

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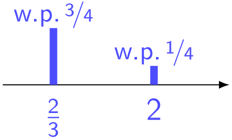

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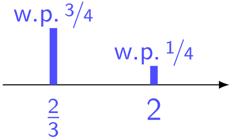

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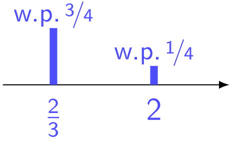

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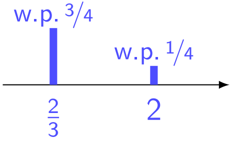

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$$= \mathop{\mathbb{E}}_{x \leftarrow Q} \max\left(1 - \frac{P(x)}{Q(x)}, 0\right) = \frac{1}{2} \mathop{\mathbb{E}}_{x \leftarrow Q}\left|1 - \frac{P(x)}{Q(x)}\right|$$

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\Delta_{\mathrm{TV}}(P, Q) &= \sum_x \max(Q(x) - P(x), 0) &&= \frac{1}{2} \sum_x |Q(x) - P(x)| \\
&= \mathop{\mathbb{E}}_{x \leftarrow Q} \max\left(1 - \frac{P(x)}{Q(x)}, 0\right) &&= \frac{1}{2} \mathop{\mathbb{E}}_{x \leftarrow Q} \left|1 - \frac{P(x)}{Q(x)}\right| \\
&= \mathbb{E}\big[\max(1 - R, 0)\big] &&= \frac{1}{2} \mathbb{E}\,|1 - R|
\end{aligned}$$

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KL divergence $= \mathbb{E}\left[R \log R\right]$

Rényi divergence $= \frac{1}{\alpha - 1}\log\left(\mathbb{E}\left[R^{\alpha}\right]\right)$

χ^2 divergence $= \mathbb{E}\left[(1 - R)^2\right]$

f-divergence $= \mathbb{E}\left[f(R)\right]$

Product Distributions

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$$\underset{R}{\overset{(P\|Q)}{\downarrow}} = \frac{P(x_1, \ldots, x_n)}{Q(x_1, \ldots, x_n)} \text{ where } (x_1, \ldots, x_n) \leftarrow Q$$

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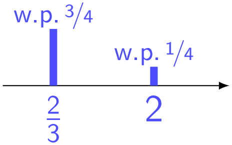
E.g. $R_1 =$



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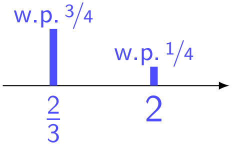

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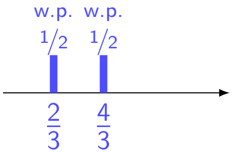

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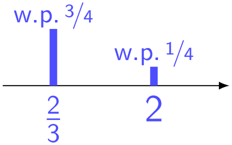

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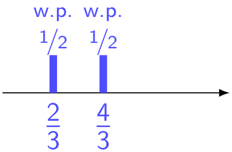

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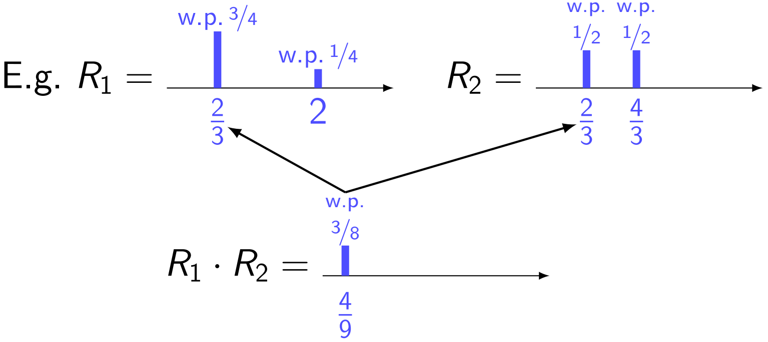

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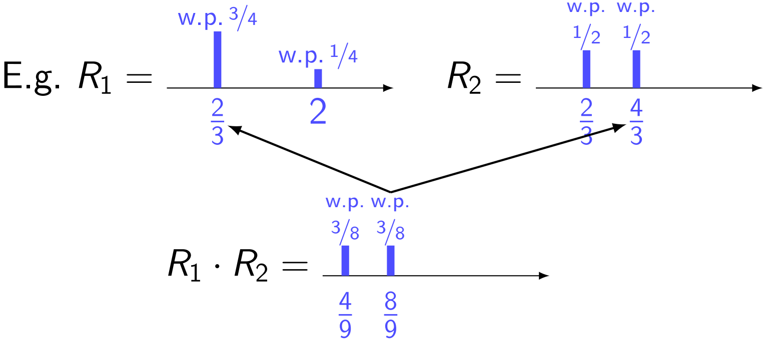

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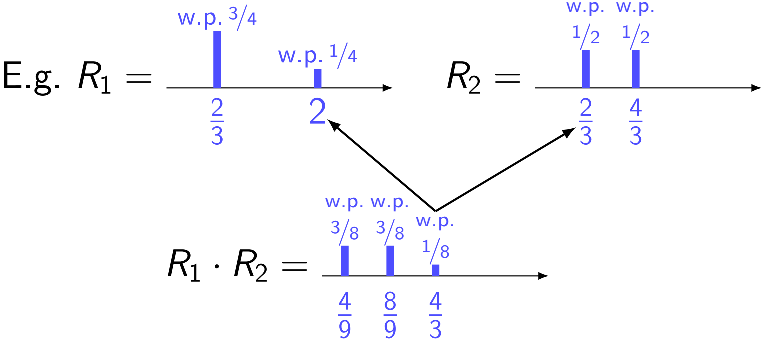

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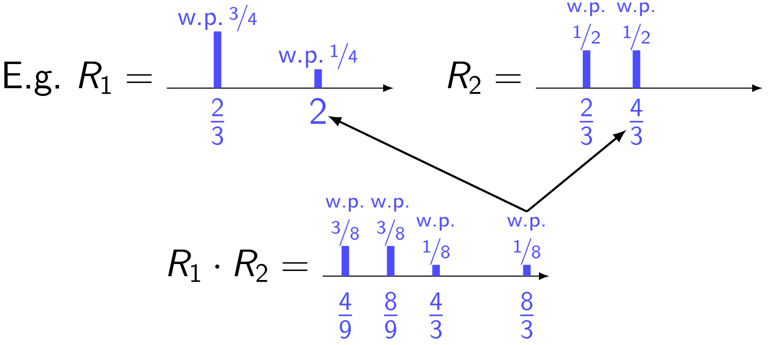

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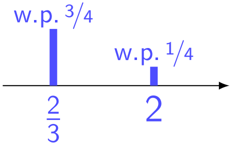
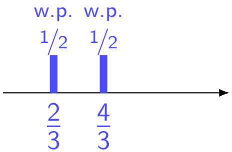
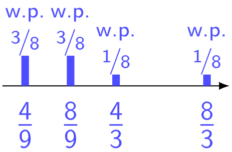
E.g. $R_1 =$ 

$R_2 =$ 

$R_1 \cdot R_2 =$ 

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Naïve Algorithm:

- Compute $R_1 \cdot R_2$

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Naïve Algorithm:

- Compute $R_1 \cdot R_2$
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- Compute $R_1 \cdot R_2$
- Compute $R_1 \cdot R_2 \cdot R_3$

$\vdots$

- Compute $R_1 \cdot R_2 \cdot \cdots \cdot R_n$

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$$\text{Ratio} \quad \underset{(P\|Q)}{\underset{\uparrow}{R}} = \underset{(P_1\|Q_1)}{\underset{\uparrow}{R_1}} \cdot \underset{(P_2\|Q_2)}{\underset{\uparrow}{R_2}} \ldots \underset{(P_n\|Q_n)}{\underset{\uparrow}{R_n}}$$

Naïve Algorithm:

- Compute $R_1 \cdot R_2$
- Compute $R_1 \cdot R_2 \cdot R_3$

$\vdots$

- Compute $R_1 \cdot R_2 \cdot \cdots \cdot R_n$
- Compute Δ_{TV}

Product Distributions

$$P = P_1 \times P_2 \times \cdots \times P_n$$

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up to 2^n (boolean domain)

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up to q^n (size-q domain)

Our algorithm (informal):

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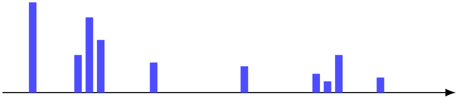
- $\vdots$
- Simplify it as $\tilde{R}_{1:n}$ // so that $\tilde{R}_{1:n} \approx R_1 \cdot R_2 \cdot \ldots R_n$
- Estimate Δ_{TV}

Simplify the Ratio

Ratio $R =$ 

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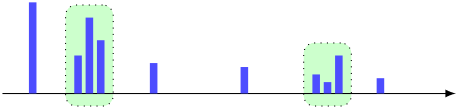
Ratio $R =$ 

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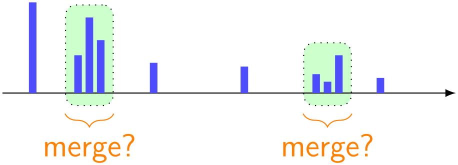
Ratio $R =$ 

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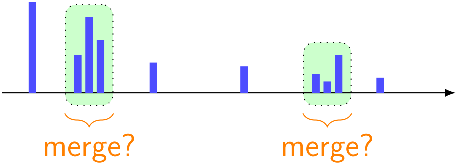
Ratio $R =$ 

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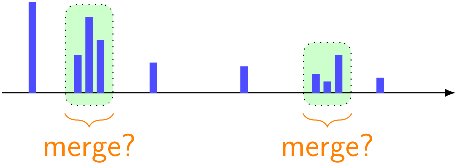
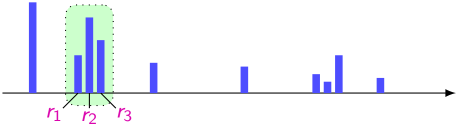
Ratio $R =$ 

- How to merge?
- How large is the error?

Simplify the Ratio

Ratio $R =$ 

$$R = \begin{cases} r_1, & \text{w.p. } q_1 \\ r_2, & \text{w.p. } q_2 \\ r_3, & \text{w.p. } q_3 \\ \text{the rest} & \end{cases}$$

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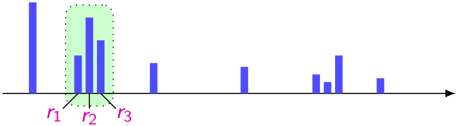
Ratio $R =$ 

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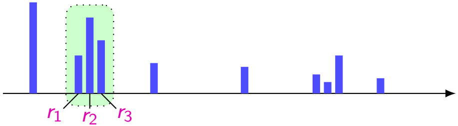
Ratio $R =$ 

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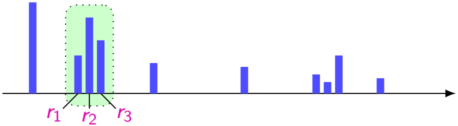
Ratio $R =$ 

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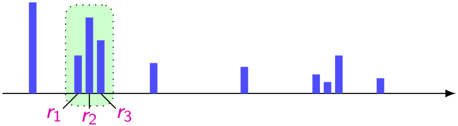
Ratio $R =$ 

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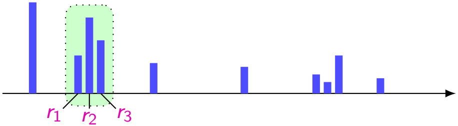

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NEXT: Prove that $\tilde{R} \approx R$

Distance between Ratios

Given two ratios $R, \tilde{R}$

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Distance between Ratios

Given two ratios $R, \tilde{R}$ such that $R \equiv (P \| Q)$, $\tilde{R} \equiv (\tilde{P} \| \tilde{Q})$,

the **minimum total variation distance**

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Hi Prof. Yury,
I found an interesting **metric**.
Δ_{MTV} is defined as ...

sent from PKU

me

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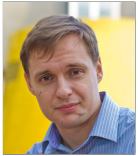

If you check LeCam's textbook,
there is a notion called ...

sent from MIT

Distance between Ratios

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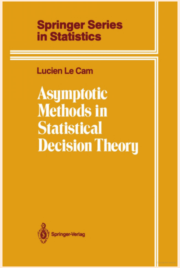

Similar metric from decision theory

Deficiency$(R, \tilde{R})$

$$= \min_{\substack{R \equiv (P \| Q) \\ \tilde{R} \equiv (\tilde{P} \| \tilde{Q})}} \max\left(\Delta_{\mathrm{TV}}(P, \tilde{P}), \Delta_{\mathrm{TV}}(Q, \tilde{Q}) \right)$$

Prove that $\tilde{R} \approx R$

$$R = \begin{cases} r_1, & \text{w.p. } q_1 \\ r_2, & \text{w.p. } q_2 \\ r_3, & \text{w.p. } q_3 \\ \text{the rest} \end{cases} \qquad P = \begin{cases} \boxed{1}, & \text{w.p. } r_1 q_1 \\ \boxed{2}, & \text{w.p. } r_2 q_2 \\ \boxed{3}, & \text{w.p. } r_3 q_3 \\ \text{the rest} \end{cases} \qquad Q = \begin{cases} \boxed{1}, & \text{w.p. } q_1 \\ \boxed{2}, & \text{w.p. } q_2 \\ \boxed{3}, & \text{w.p. } q_3 \\ \text{the rest} \end{cases}$$

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Prove that $\tilde{R} \approx R$

Condition: r_1, r_2, r_3 are close.

$$R = \begin{cases} r_1, & \text{w.p. } q_1 \\ r_2, & \text{w.p. } q_2 \\ r_3, & \text{w.p. } q_3 \\ \text{the rest} \end{cases} \quad P = \begin{cases} \boxed{1}, & \text{w.p. } r_1 q_1 \\ \boxed{2}, & \text{w.p. } r_2 q_2 \\ \boxed{3}, & \text{w.p. } r_3 q_3 \\ \text{the rest} \end{cases} \quad Q = \begin{cases} \boxed{1}, & \text{w.p. } q_1 \\ \boxed{2}, & \text{w.p. } q_2 \\ \boxed{3}, & \text{w.p. } q_3 \\ \text{the rest} \end{cases}$$

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Prove that $\tilde{R} \approx R$

Condition: r_1, r_2, r_3 are close. Let $r^* = \dfrac{r_1 q_1 + r_2 q_2 + r_3 q_3}{q_1 + q_2 + q_3}$

$$R = \begin{cases} r_1, & \text{w.p. } q_1 \\ r_2, & \text{w.p. } q_2 \\ r_3, & \text{w.p. } q_3 \\ \text{the rest} \end{cases} \qquad P = \begin{cases} \boxed{1}, & \text{w.p. } r_1 q_1 \\ \boxed{2}, & \text{w.p. } r_2 q_2 \\ \boxed{3}, & \text{w.p. } r_3 q_3 \\ \text{the rest} \end{cases} \qquad Q = \begin{cases} \boxed{1}, & \text{w.p. } q_1 \\ \boxed{2}, & \text{w.p. } q_2 \\ \boxed{3}, & \text{w.p. } q_3 \\ \text{the rest} \end{cases}$$

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$$\tilde{R} = \begin{cases} r^*, \text{ w.p. } q_1 + q_2 + q_3 \\ \text{the rest} \end{cases} \quad \tilde{P} = \begin{cases} \boxed{1}, & \text{w.p. } r^* q_1 \\ \boxed{2}, & \text{w.p. } r^* q_2 \\ \boxed{3}, & \text{w.p. } r^* q_3 \\ \text{the rest} & \end{cases} \qquad \tilde{Q} = Q$$

Prove that $\tilde{R} \approx R$

Condition: r_1, r_2, r_3 are close. Let $r^* = \dfrac{r_1 q_1 + r_2 q_2 + r_3 q_3}{q_1 + q_2 + q_3}$

$$R = \begin{cases} r_1, & \text{w.p. } q_1 \\ r_2, & \text{w.p. } q_2 \\ r_3, & \text{w.p. } q_3 \\ \text{the rest} & \end{cases} \qquad P = \begin{cases} \boxed{1}, & \text{w.p. } r_1 q_1 \\ \boxed{2}, & \text{w.p. } r_2 q_2 \\ \boxed{3}, & \text{w.p. } r_3 q_3 \\ \text{the rest} & \end{cases} \qquad Q = \begin{cases} \boxed{1}, & \text{w.p. } q_1 \\ \boxed{2}, & \text{w.p. } q_2 \\ \boxed{3}, & \text{w.p. } q_3 \\ \text{the rest} & \end{cases}$$

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$$\Delta_{\mathsf{MTV}}(R, \tilde{R}) \leq \Delta_{\mathsf{TV}}(P, \tilde{P}) + \Delta_{\mathsf{TV}}(Q, \tilde{Q})$$

Prove that $\tilde{R} \approx R$

Condition: r_1, r_2, r_3 are close. Let $r^* = \dfrac{r_1 q_1 + r_2 q_2 + r_3 q_3}{q_1 + q_2 + q_3}$

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$$\Delta_{\mathrm{MTV}}(R, \tilde{R}) \le \Delta_{\mathrm{TV}}(P, \tilde{P}) \le |r_1 - r^*| q_1 + |r_2 - r^*| q_2 + |r_3 - r^*| q_3$$

Prove that $\tilde{R} \approx R$

Merge all masses in $[a, b)$ for $0 \le a < b < 1$,
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Prove that $\tilde{R} \approx R$

Merge all masses in $[a, b)$ for $0 \leq a < b < 1$, the result is $\tilde{R}$.

Condition: $\frac{1-a}{1-b} \leq 1 + \varepsilon$

Say the masses are on $r_1, r_2, r_3 \in [a, b)$. Let $r^* = \dfrac{r_1 q_1 + r_2 q_2 + r_3 q_3}{q_1 + q_2 + q_3}$.

$$\Delta_{\mathrm{MTV}}(R, \tilde{R}) \leq \Delta_{\mathrm{TV}}(P, \tilde{P}) \leq |r_1 - r^*| q_1 + |r_2 - r^*| q_2 + |r_3 - r^*| q_3$$

Prove that $\tilde{R} \approx R$

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Condition: $\frac{1-a}{1-b} \leq 1 + \varepsilon \implies |r_i - r^*| \leq \varepsilon(1 - r_i)$

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$$\begin{aligned}\Delta_{\mathrm{MTV}}(R, \tilde{R}) \leq \Delta_{\mathrm{TV}}(P, \tilde{P}) &\leq |r_1 - r^*| q_1 + |r_2 - r^*| q_2 + |r_3 - r^*| q_3 \\ &\leq \varepsilon(1 - r_1) q_1 + \varepsilon(1 - r_2) q_2 + \varepsilon(1 - r_3) q_3\end{aligned}$$

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$$
\begin{aligned}
\Delta_{\mathrm{MTV}}(R, \tilde{R}) \leq \Delta_{\mathrm{TV}}(P, \tilde{P}) &\leq |r_1 - r^*| q_1 + |r_2 - r^*| q_2 + |r_3 - r^*| q_3 \\
&\leq \varepsilon(1 - r_1) q_1 + \varepsilon(1 - r_2) q_2 + \varepsilon(1 - r_3) q_3 \\
&\leq \varepsilon(1 - r_1) \Pr[R{=}r_1] + \varepsilon(1 - r_2) \Pr[R{=}r_2] + \varepsilon(1 - r_3) \Pr[R{=}r_3]
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&\leq \varepsilon(1 - r_1)q_1 + \varepsilon(1 - r_2)q_2 + \varepsilon(1 - r_3)q_3 \\
&\leq \varepsilon(1 - r_1)\Pr[R{=}r_1] + \varepsilon(1 - r_2)\Pr[R{=}r_2] + \varepsilon(1 - r_3)\Pr[R{=}r_3] \\
&= \varepsilon \sum_{a \leq r < b} (1 - r)\Pr[R = r]
\end{aligned}
$$

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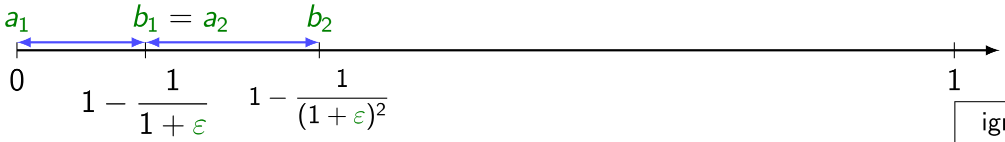

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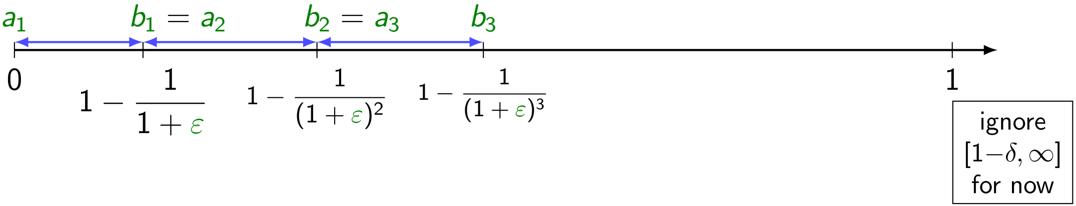

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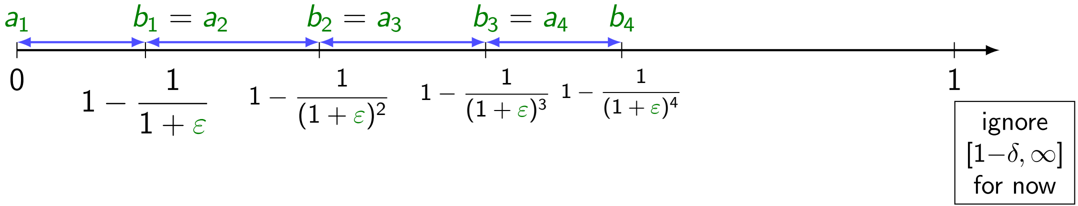

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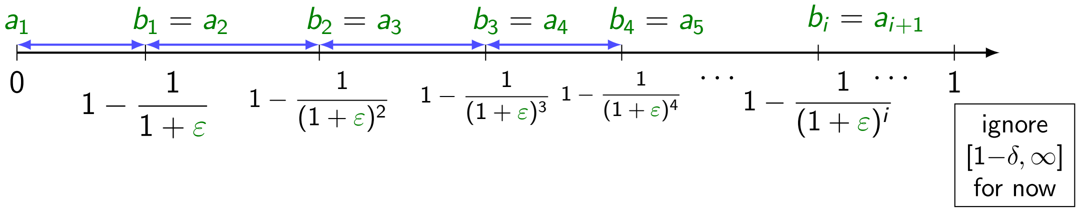

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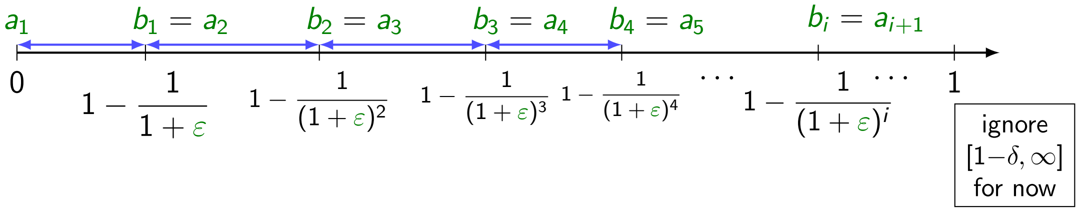

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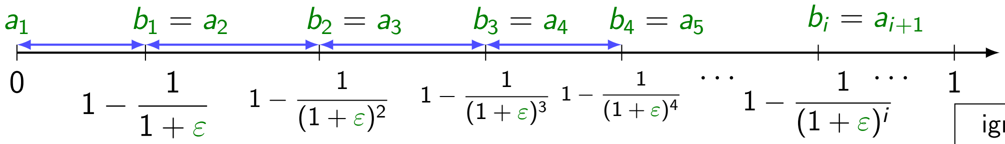

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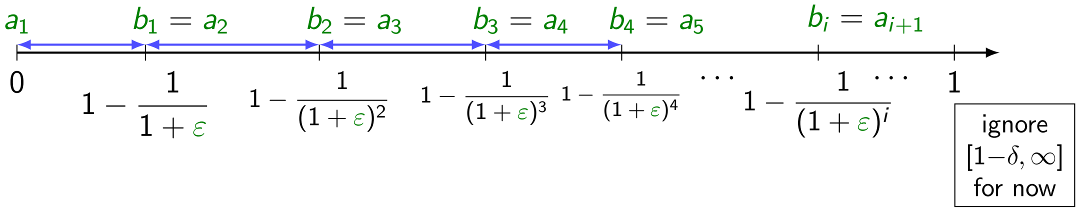

$$\Delta_{\mathrm{MTV}}(R, \tilde{R}) \leq \varepsilon \sum_{a_i \leq r < b_i} (1-r) \Pr[R = r]$$

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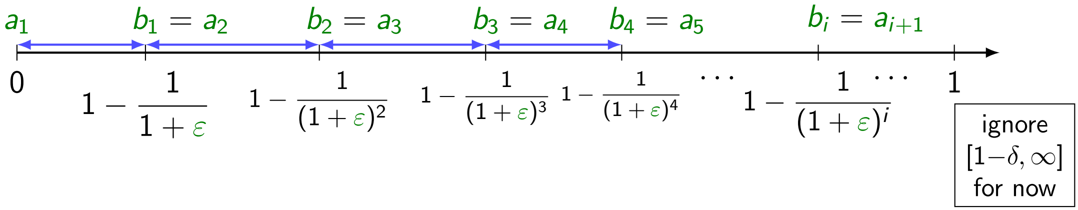

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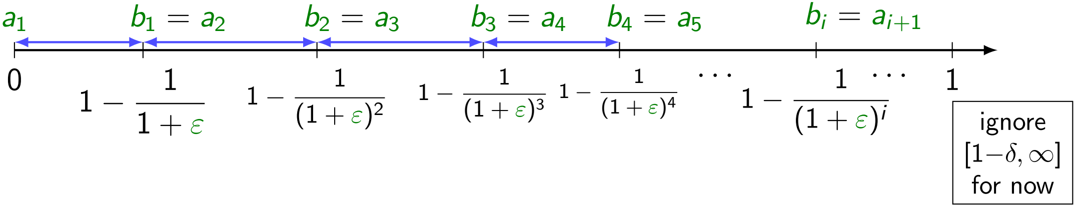

$$\Delta_{\mathrm{MTV}}(R, \tilde{R}) \leq \varepsilon \sum_i \sum_{a_i \leq r < b_i} (1 - r) \Pr[R = r]$$

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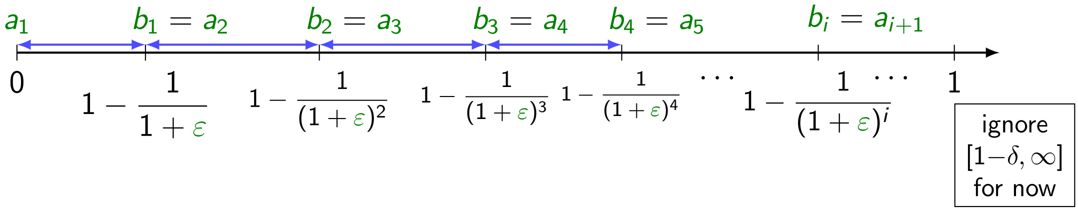

$$\Delta_{\mathrm{MTV}}(R, \tilde{R}) \leq \varepsilon \sum_i \sum_{a_i \leq r < b_i} (1 - r) \Pr[R = r] \leq \varepsilon \sum_{r<1} (1 - r) \Pr[R = r]$$

Prove that $\tilde{R} \approx R$

Algorithm: Let $a_{i+1} = b_i = 1 - 1/(1+\varepsilon)^i$
For each i, Merge all masses in $[a_i, b_i)$
The result is $\tilde{R}$

$$\Delta_{\mathrm{MTV}}(R, \tilde{R}) \leq \varepsilon \sum_i \sum_{a_i \leq r < b_i} (1 - r) \Pr[R = r] \leq \varepsilon \sum_{r<1} (1 - r) \Pr[R = r]$$

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$$\Delta_{\mathrm{TV}}(R) := \Delta_{\mathrm{TV}}(P, Q)$$

$$\Delta_{\mathrm{MTV}}(R, \tilde{R}) \leq \varepsilon \sum_i \sum_{a_i \leq r < b_i} (1 - r) \Pr[R = r] \leq \varepsilon \sum_{r<1} (1 - r) \Pr[R = r]$$

Prove that $\tilde{R} \approx R$

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The result is $\tilde{R}$

$$\Delta_{\mathrm{TV}}(R) := \Delta_{\mathrm{TV}}(P, Q) = \mathop{\mathbb{E}}_{r \leftarrow R}\left[\max(1 - r, 0)\right]$$

$$\Delta_{\mathrm{MTV}}(R, \tilde{R}) \leq \varepsilon \sum_i \sum_{a_i \leq r < b_i} (1 - r)\Pr[R = r] \leq \varepsilon \sum_{r < 1} (1 - r)\Pr[R = r]$$

Prove that $\tilde{R} \approx R$

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For each i, Merge all masses in $[a_i, b_i)$

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$$\Delta_{\mathrm{MTV}}(R, \tilde{R}) \leq \varepsilon \sum_i \sum_{a_i \leq r < b_i} (1 - r) \Pr[R = r] \leq \varepsilon \sum_{r<1} (1 - r) \Pr[R = r]$$

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Prove that $\tilde{R} \approx R$

Algorithm: Let $a_{i+1} = b_i = 1 - 1/(1+\varepsilon)^i$

For each i, Merge all masses in $[a_i, b_i)$

The result is $\tilde{R}$

$$\Delta_{\mathrm{MTV}}(R, \tilde{R}) \leq \varepsilon\, \Delta_{\mathrm{TV}}(R)$$

Prove that $\tilde{R} \approx R$

Algorithm: Let $a_{i+1} = b_i = 1 - 1/(1+\varepsilon)^i$

For each i, Merge all masses in $[a_i, b_i)$

For each i, Merge all masses in $(1/b_i, 1/a_i]$

The result is $\tilde{R}$

$$\Delta_{\text{MTV}}(R, \tilde{R}) \leq \varepsilon\, \Delta_{\text{TV}}(R)$$

Prove that $\tilde{R} \approx R$

Algorithm: Let $a_{i+1} = b_i = 1 - 1/(1+\varepsilon)^i$

For each i, Merge all masses in $[a_i, b_i)$

For each i, Merge all masses in $(1/b_i, 1/a_i]$

The result is $\tilde{R}$

$$\Delta_{\mathrm{MTV}}(R, \tilde{R}) \leq \varepsilon\, \Delta_{\mathrm{TV}}(R) + \varepsilon\, \Delta_{\mathrm{TV}}(R)$$

Prove that $\tilde{R} \approx R$

Algorithm: Let $a_{i+1} = b_i = 1 - 1/(1+\varepsilon)^i$

Merge all masses in $[1-\delta, 1+\delta]$ for $\delta < \varepsilon\,\Delta_{\mathrm{TV}}$

For each i, Merge all masses in $[a_i, b_i)$

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For each i, Merge all masses in $(1/b_i, 1/a_i]$

The result is $\tilde{R}$

$$\Delta_{\mathrm{MTV}}(R, \tilde{R}) \leq \varepsilon\,\Delta_{\mathrm{TV}}(R) + \varepsilon\,\Delta_{\mathrm{TV}}(R) + 2\varepsilon\,\Delta_{\mathrm{TV}}(R)$$

Our algorithm:

- Compute $R_1 \cdot R_2$

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- Compute $\tilde{R}_{1:2} \cdot R_3$
- **Sparsify** it as $\tilde{R}_{1:3}$ // s.t. $\Delta_{\mathrm{MTV}}(\tilde{R}_{1:3}, \tilde{R}_{1:2} \cdot R_3) \leq \varepsilon\, \Delta_{\mathrm{TV}}$

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- Compute $\tilde{R}_{1:2} \cdot R_3$
- **Sparsify** it as $\tilde{R}_{1:3}$ // s.t. $\Delta_{\mathrm{MTV}}(\tilde{R}_{1:3}, \tilde{R}_{1:2} \cdot R_3) \leq \varepsilon\, \Delta_{\mathrm{TV}}$
- Compute $\tilde{R}_{1:3} \cdot R_4$
- $\vdots$
- **Sparsify** it as $\tilde{R}_{1:n}$ // s.t. $\Delta_{\mathrm{MTV}}(\tilde{R}_{1:n}, \tilde{R}_{1:n-1} \cdot R_n) \leq \varepsilon\, \Delta_{\mathrm{TV}}$

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- Compute $\tilde{R}_{1:2} \cdot R_3$
- **Sparsify** it as $\tilde{R}_{1:3}$ // s.t. $\Delta_{\mathrm{MTV}}(\tilde{R}_{1:3}, \tilde{R}_{1:2} \cdot R_3) \leq \varepsilon\, \Delta_{\mathrm{TV}}$
- Compute $\tilde{R}_{1:3} \cdot R_4$
- $\vdots$
- **Sparsify** it as $\tilde{R}_{1:n}$ // s.t. $\Delta_{\mathrm{MTV}}(\tilde{R}_{1:n}, \tilde{R}_{1:n-1} \cdot R_n) \leq \varepsilon\, \Delta_{\mathrm{TV}}$

Triangular inequality: $\Delta_{\mathrm{MTV}}(\tilde{R}_{1:n}, R_1 \cdot R_2 \cdots R_n) \leq n\varepsilon\, \Delta_{\mathrm{TV}}$

Our algorithm:

- Compute $R_1 \cdot R_2$
- **Sparsify** it as $\tilde{R}_{1:2}$ // s.t. $\Delta_{\mathrm{MTV}}(\tilde{R}_{1:2}, R_1 \cdot R_2) \leq \varepsilon\, \Delta_{\mathrm{TV}}$
- Compute $\tilde{R}_{1:2} \cdot R_3$
- **Sparsify** it as $\tilde{R}_{1:3}$ // s.t. $\Delta_{\mathrm{MTV}}(\tilde{R}_{1:3}, \tilde{R}_{1:2} \cdot R_3) \leq \varepsilon\, \Delta_{\mathrm{TV}}$
- Compute $\tilde{R}_{1:3} \cdot R_4$
- $\vdots$
- **Sparsify** it as $\tilde{R}_{1:n}$ // s.t. $\Delta_{\mathrm{MTV}}(\tilde{R}_{1:n}, \tilde{R}_{1:n-1} \cdot R_n) \leq \varepsilon\, \Delta_{\mathrm{TV}}$

  Triangular inequality: $\Delta_{\mathrm{MTV}}(\tilde{R}_{1:n}, R_1 \cdot R_2 \cdots R_n) \leq n\varepsilon\, \Delta_{\mathrm{TV}}$
- Estimate Δ_{TV} // err $\leq n\varepsilon\, \Delta_{\mathrm{TV}}$

Problem: Given two **product** distributions P, Q
Approximate the total variation distance

This work: Feng-Liu-Liu

Deterministic algorithm, in time $O(qn^2\varepsilon^{-1}\log\frac{n}{\varepsilon\,\Delta_{\mathsf{TV}}(P,Q)})$

- q: domain size
- ε: relative error

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Problem: Given two **Markov chains** P, Q
Approximate the total variation distance

Problem: Given two **product** distributions P, Q

Approximate the total variation distance

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Deterministic algorithm, in time $O(qn^2\varepsilon^{-1}\log\frac{n}{\varepsilon\,\Delta_{\mathrm{TV}}(P,Q)})$

- q: domain size
- ε: relative error

Problem: Given two **Markov chains** P, Q

Approximate the total variation distance

Deterministic algorithm, in time $O(q^2n^2\varepsilon^{-1}\log\frac{n}{\varepsilon\,\Delta_{\mathrm{TV}}(P,Q)})$