Breaking the Circuit-Size Barrier in Secret Sharing

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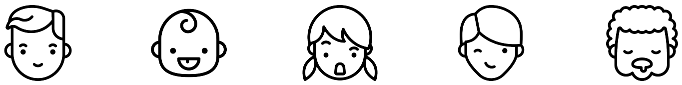
June 27, 2018

Secret Sharing [Blakley'79,Shamir'79,Ito-Saito-Nishizeki'87]

Secret



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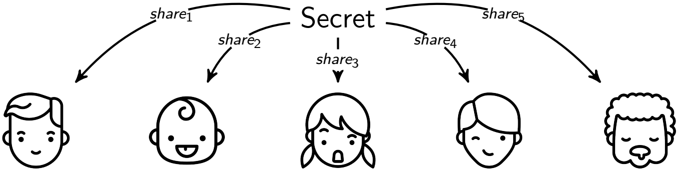

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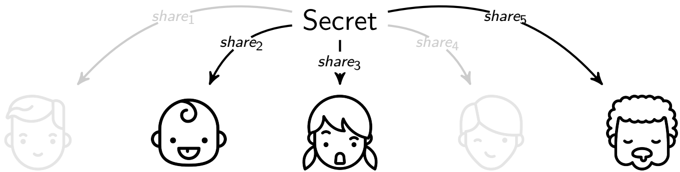

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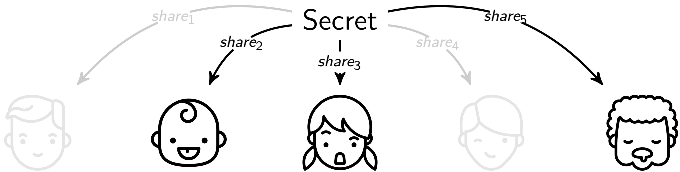

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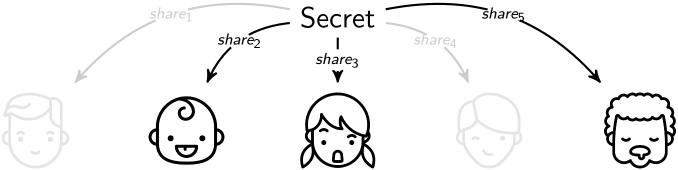

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Any subset of $\geq k$ participants can recover the secret.
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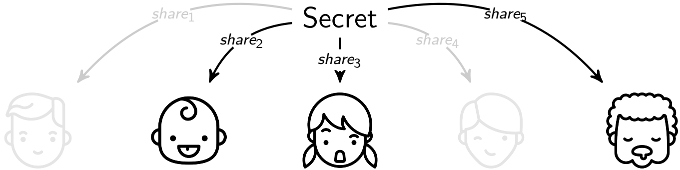

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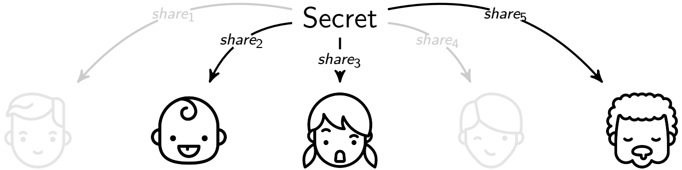

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A General Secret Sharing Scheme [Benaloh-Leichter'88]

F is computed by some monotone formula

- Generate a tag for each wire
  - Output wire tag: the secret s
  - AND gate: additively share the output wire tag
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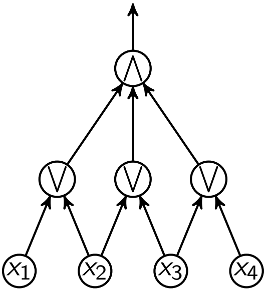
- The i-th participant's share: all tags of its input wires



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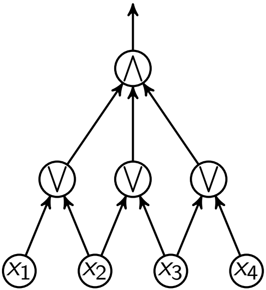

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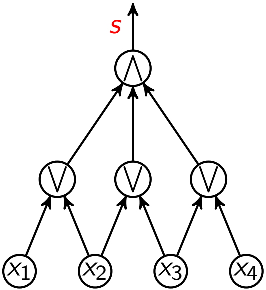

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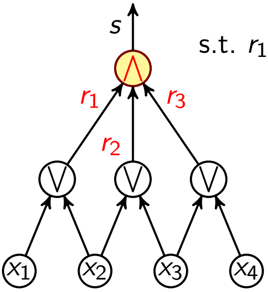

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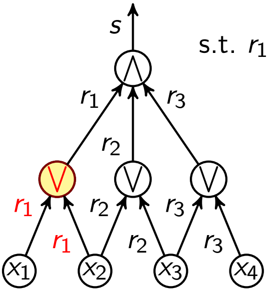

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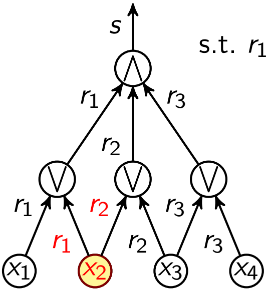

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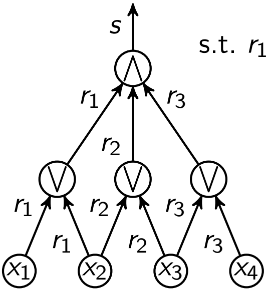

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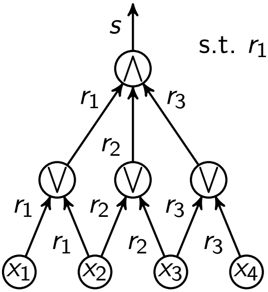

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Key Complexity Measure: Total Share Size

Upper Bounds

Share size $= O$(monotone formula size) [Benaloh-Leichter'88]

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Share size $= O$(monotone formula size) [Benaloh-Leichter'88]

Share size $= O$(monotone span program size) [Karchmer-Wigderson'93]

Key Complexity Measure: Total Share Size

Upper Bounds

Share size $= O(\text{monotone formula size}) \leq \frac{2^n}{\text{poly}(n)}$.

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Exists an explicit F s.t. total share size $= \tilde{\Omega}(n^2)$. [Csirmaz'97]

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Can we do better?

30^{+}-year-old open problem

Our Results

Yes, we can!

Theorem 1

Every monotone F has a secret sharing scheme with share size $2^{0.994n}$.

Key Complexity Measure: Total Share Size

Upper Bounds: *Linear* Secret Sharing

Share size $= O(\text{monotone formula size}) \leq \frac{2^n}{\text{poly}(n)}$.

Share size $= \Theta(\text{monotone span program size}) \leq \frac{2^n}{\text{poly}(n)}$.

Lower Bounds: *Linear* Secret Sharing

Exists $\{F_n\}$ s.t. total share size $= \tilde{\Omega}(2^{n/2})$.

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Key Complexity Measure: Total Share Size

Upper Bounds: *Linear* Secret Sharing

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($2^{\Omega(n)}$ for an explicit $\{F_n\}$ [Pitassi-Robere'18])

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Theorem 2

Every monotone F has a *linear* secret sharing with share size $2^{0.999n}$.

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Corollary

Every monotone F has a monotone span program of size $2^{0.999n}$.

Our Approach

Every monotone F can be computed by a monotone formula s.t.

Prop. I

Prop. II

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Prop. II

$$\text{Formula size} \gtrsim \log(\#\text{Monotone Functions}) \geq \frac{2^n}{\text{poly}(n)}$$

Our Approach

Every monotone F can be computed by a monotone formula s.t.

Prop. I has size $2^{0.994n}$

Prop. II

$$\text{Formula size} \times \log(\#\text{Base Gates}) \geq \log(\#\text{Monotone Functions}) \geq \frac{2^n}{\text{poly}(n)}$$

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Every monotone F can be computed by a monotone formula s.t.

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$$\text{Formula size} \times \log(\#\text{Base Gates}) \geq \log(\#\text{Monotone Functions}) \geq \frac{2^n}{\text{poly}(n)}$$

$\implies$ Requires $2^{\tilde{\Omega}(2^n)}$ gates in formula basis.

Our Approach

Every monotone F can be computed by a monotone formula s.t.

Prop. I has size $2^{0.994n}$ using an extended basis of $2^{\tilde{\Omega}(2^n)}$ gates

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Every monotone F can be computed by a monotone formula s.t.

- Prop. I has size $2^{0.994n}$ using an extended basis of $2^{\tilde{\Omega}(2^n)}$ gates
- Prop. II every gate in the basis is a monotone function that has an efficient secret sharing scheme

Base gates [Liu-Vaikuntanathan-Wee'18]

We define **slice functions**, there are $2^{\binom{n}{n/2}}$ of them and they have secret scharing scheme with share size $2^{\tilde{O}(\sqrt{n})}$.

Our Approach

Slice Functions

all F such that
$\|x\| > n/2 \implies F(x) = 1$
$\|x\| < n/2 \implies F(x) = 0$

$\#\text{functions} = 2^{\binom{n}{n/2}}$

Share size $= 2^{\tilde{O}(\sqrt{n})}$

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Monotone Functions

all monotone F

Share size $= 2^{0.994n}$

monotone formula
size: $2^{0.994n}$
depth: constant
gates: $\wedge, \vee$, slice functions

Our Approach

Slice Functions	*Fat-Slice Functions*	*Monotone Functions*
all F such that $\|x\| > n/2 \implies F(x) = 1$ $\|x\| < n/2 \implies F(x) = 0$	all F such that $\|x\| > .51n \Rightarrow F(x) = 1$ $\|x\| < .49n \Rightarrow F(x) = 0$	all monotone F
$\#\text{functions} = 2^{\binom{n}{n/2}}$		
Share size $= 2^{\tilde{O}(\sqrt{n})}$	Share size $= 2^{(1-c)n}$	Share size $= 2^{0.994n}$

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size: $2^{(1-c')n}$
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Fat-Slice Functions

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Fat-Slice Functions $\implies$ All Monotone Functions

Let F be any monotone function.
Define $F_{\text{bot}}, F_{\text{mid}}, F_{\text{top}}$ as the following:

$$F_{\text{bot}}(x) = \bigvee_{\substack{y \text{ s.t.} \\ \|y\|<.49n \\ F(y)=1}} \mathbb{1}_{x \geq y} = \bigvee_{\substack{y \text{ s.t.} \\ \|y\|<.49n \\ F(y)=1}} \bigwedge_{i, y_i=1} x_i$$

$$F_{\text{mid}}(x) = \begin{cases} 0, & \text{if } \|x\| < .49n \\ F(x), & \text{if } \|x\| \approx .5n \\ 1, & \text{if } \|x\| > .51n \end{cases}$$

F_{mid} is a fat-slice function.

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$$F_{\text{bot}}(x) = \bigvee_{\substack{y \text{ s.t.} \\ \|y\|<.49n \\ F(y)=1}} \mathbb{1}_{x \geq y} = \bigvee_{\substack{y \text{ s.t.} \\ \|y\|<.49n \\ F(y)=1}} \bigwedge_{i, y_i=1} x_i$$

$$F_{\text{mid}}(x) = \begin{cases} 0, & \text{if } \|x\| < .49n \\ F(x), & \text{if } \|x\| \approx .5n \\ 1, & \text{if } \|x\| > .51n \end{cases}$$

F_{mid} is a fat-slice function.
Share size $= 2^{(1-c)n}$

$$F_{\text{top}}(x) = \bigwedge_{\substack{y \text{ s.t.} \\ \|y\|>.51n \\ F(y)=0}} \mathbb{1}_{x \not\leq y} = \bigwedge_{\substack{y \text{ s.t.} \\ \|y\|>.51n \\ F(y)=0}} \bigvee_{i, y_i=0} x_i$$

$F_{\text{bot}}, F_{\text{top}}$ has monotone formula of size $2^{h(.49)\cdot n} = 2^{(1-c')n}$
$\Longrightarrow$ Share size $= 2^{(1-c')n}$

Fat-Slice Functions $\implies$ All Monotone Functions

Let F be any monotone function.
Define $F_{\text{bot}}, F_{\text{mid}}, F_{\text{top}}$ such that:

<table>
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- $F(x) = \text{Majority}(F_{\text{bot}}(x), F_{\text{mid}}(x), F_{\text{top}}(x))$

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- $F(x) = (F_{\text{bot}}(x) \vee F_{\text{mid}}(x)) \wedge F_{\text{top}}(x)$

Fat-Slice Functions $\implies$ All Monotone Functions

Let F be any monotone function.
Define $F_{\text{bot}}, F_{\text{mid}}, F_{\text{top}}$ such that:

- F_{mid} lays in "a fat slice" $[49\%, 51\%]$
  $\implies$ Share size of $F_{\text{mid}} = 2^{(1-c)n}$
- $F_{\text{bot}}, F_{\text{top}}$ computed by size-$2^{h(.49)\cdot n}$ formula
  $\implies$ Share size of $F_{\text{bot}}, F_{\text{top}} = 2^{(1-c')n}$
- $F(x) = F_{\text{bot}}(x) \vee F_{\text{mid}}(x) \wedge F_{\text{top}}(x)$
  $\implies$ Share size of $F = 2^{(1-c)n} + 2 \cdot 2^{(1-c')n}$
  $= O(2^{\max(1-c, 1-c')n})$

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Fat-Slice Functions $\implies$ All Monotone Functions

Let F be any monotone function.
Define $F_{\text{bot}}, F_{\text{mid}}, F_{\text{top}}$ such that:

- F_{mid} lays in "a fatter slice" $[40\%, 60\%]$
  $\implies$ Share size of $F_{\text{mid}} = 2^{(1-c)n}$
- $F_{\text{bot}}, F_{\text{top}}$ computed by size-$2^{h(.49)\cdot n}$ formula
  $\implies$ Share size of $F_{\text{bot}}, F_{\text{top}} = 2^{(1-c')n}$
- $F(x) = F_{\text{bot}}(x) \vee F_{\text{mid}}(x) \wedge F_{\text{top}}(x)$
  $\implies$ Share size of F $= 2^{(1-c)n} + 2 \cdot 2^{(1-c')n}$
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Fat-Slice Functions $\implies$ All Monotone Functions

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Define $F_{\text{bot}}, F_{\text{mid}}, F_{\text{top}}$ such that:

- F_{mid} lays in "a fatter slice" $[40\%, 60\%]$
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- $F(x) = F_{\text{bot}}(x) \vee F_{\text{mid}}(x) \wedge F_{\text{top}}(x)$
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Fat-Slice Functions $\Longrightarrow$ All Monotone Functions

Let F be any monotone function.
Define $F_{\text{bot}}, F_{\text{mid}}, F_{\text{top}}$ such that:

- F_{mid} lays in "a fatter slice" $[40\%, 60\%]$
  $\Longrightarrow$ Share size of $F_{\text{mid}} = 2^{(1-c)n}$ increase$\uparrow\uparrow$
- $F_{\text{bot}}, F_{\text{top}}$ computed by size-$2^{h(.4)\cdot n}$ formula
  $\Longrightarrow$ Share size of $F_{\text{bot}}, F_{\text{top}} = 2^{(1-c')n}$
- $F(x) = F_{\text{bot}}(x) \vee F_{\text{mid}}(x) \wedge F_{\text{top}}(x)$
  $\Longrightarrow$ Share size of $F = 2^{(1-c)n} + 2 \cdot 2^{(1-c')n}$
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Fat-Slice Functions $\implies$ All Monotone Functions

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  $\implies$ Share size of $F_{\text{bot}}, F_{\text{top}} = 2^{(1-c')n}$ decrease↓↓
- $F(x) = F_{\text{bot}}(x) \vee F_{\text{mid}}(x) \wedge F_{\text{top}}(x)$
  $\implies$ Share size of F $= 2^{(1-c)n} + 2 \cdot 2^{(1-c')n}$
  $= O(2^{\max(1-c,1-c')n})$

To Summarize

Slice Functions

all F such that

$\|x\| > n/2 \implies F(x) = 1$

$\|x\| < n/2 \implies F(x) = 0$

$\#\text{functions} = 2^{\binom{n}{n/2}}$

Share size $= 2^{\tilde{O}(\sqrt{n})}$

Fat-Slice Functions

all F such that

$\|x\| > .51n \Rightarrow F(x) = 1$

$\|x\| < .49n \Rightarrow F(x) = 0$

Share size $= 2^{(1-c')n}$

Monotone Functions

all monotone F

Share size $= 2^{(1-c)n}$

monotone formula

$F(x) = F_{\text{bot}}(x) \vee F_{\text{mid}}(x) \wedge F_{\text{top}}(x)$

To Summarize

Slice Functions

all F such that
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Fat-Slice Functions

all F such that
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Share size $= 2^{(1-c')n}$

Monotone Functions

all monotone F

Share size $= 2^{0.994n}$

monotone formula
size: $2^{0.994n}$
depth: constant
gates: $\wedge, \vee$, slice functions

To Summarize

Slice Functions	*Fat-Slice Functions*	*Monotone Functions*
all F such that $\|x\| > n/2 \implies F(x) = 1$ $\|x\| < n/2 \implies F(x) = 0$	all F such that $\|x\| > .51n \Rightarrow F(x) = 1$ $\|x\| < .49n \Rightarrow F(x) = 0$	all monotone F
$\#\text{functions} = 2^{\binom{n}{n/2}}$		
Share size $= 2^{\tilde{O}(\sqrt{n})}$	Share size $= 2^{(1-c')n}$	Share size $= 2^{0.994n}$

Previous Work [LVW'18]

monotone formula
size: $2^{0.994n}$
depth: constant
gates: $\wedge, \vee$, slice functions

Our Result

To Summarize

Slice Functions

all F such that
$\|x\| > n/2 \implies F(x) = 1$
$\|x\| < n/2 \implies F(x) = 0$

$\#\text{functions} = 2^{\binom{n}{n/2}}$

Share size $= 2^{\tilde{O}(\sqrt{n})}$

Fat-Slice Functions

all F such that
$\|x\| > .51n \Rightarrow F(x) = 1$
$\|x\| < .49n \Rightarrow F(x) = 0$

Share size $= 2^{(1-c')n}$

Monotone Functions

all monotone F

Share size $= 2^{0.994n}$

monotone formula
size: $2^{0.994n}$
depth: constant
gates: $\wedge, \vee$, slice functions

To Summarize

Slice Functions	*Fat-Slice Functions*	*Monotone Functions*
all F such that $\|x\| > n/2 \Longrightarrow F(x) = 1$ $\|x\| < n/2 \Longrightarrow F(x) = 0$	all F such that $\|x\| > .51n \Rightarrow F(x) = 1$ $\|x\| < .49n \Rightarrow F(x) = 0$	all monotone F
$\#$functions $= 2^{\binom{n}{n/2}}$		
Share size $= 2^{\tilde{O}(\sqrt{n})}$	Share size $= 2^{(1-c')n}$	Share size $= 2^{0.1n}$

monotone formula
size: $2^{0.1n}$
depth: constant
gates: $\wedge, \vee$, slice functions

Open Problem!

To Summarize

Slice Functions

all F such that
$\|x\| > n/2 \implies F(x) = 1$
$\|x\| < n/2 \implies F(x) = 0$

$\#\text{functions} = 2^{\binom{n}{n/2}}$

Share size $= 2^{\tilde{O}(\sqrt{n})}$

Fat-Slice Functions

all F such that
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Share size $= 2^{(1-c')n}$

Monotone Functions

all monotone F

Share size $= 2^{\tilde{O}(\sqrt{n})}$

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size: $2^{\tilde{O}(\sqrt{n})}$
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$\boxed{\textit{Open Problem!}}$

To Summarize (Linear Secret Sharing)

Slice Functions

all F such that
$\|x\| > n/2 \implies F(x) = 1$
$\|x\| < n/2 \implies F(x) = 0$

$\#\text{functions} = 2^{\binom{n}{n/2}}$

Share size $= \tilde{\Theta}(2^{n/2})$

Monotone Functions

all monotone F

Share size $= 2^{0.999n}$

monotone formula
size: $2^{0.999n}$
depth: constant
gates: $\wedge, \vee$, $2^{0.499} \times$slice functions

To Summarize (Linear Secret Sharing)

Slice Functions

all F such that
$\|x\| > n/2 \implies F(x) = 1$
$\|x\| < n/2 \implies F(x) = 0$

$\#\text{functions} = 2^{\binom{n}{n/2}}$

Share size $= \tilde{\Theta}(2^{n/2})$ (tight)

Monotone Functions

all monotone F

Share size $= 2^{0.999n}$

monotone formula
size: $2^{0.999n}$
depth: constant
gates: $\wedge, \vee$, $2^{0.499} \times$slice functions

To Summarize (Linear Secret Sharing)

Slice Functions

all F such that
$\|x\| > n/2 \implies F(x) = 1$
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Share size $= \tilde{\Theta}(2^{n/2})$ (tight)

Monotone Functions

all monotone F

Share size $= 2^{0.999n}$

Corollary: Monotone Span Program Complexity

Every monotone F has a monotone span program of size $2^{0.999n}$.

To Summarize

Secret sharing for any monotone function:

To Summarize

Secret sharing for any monotone function:

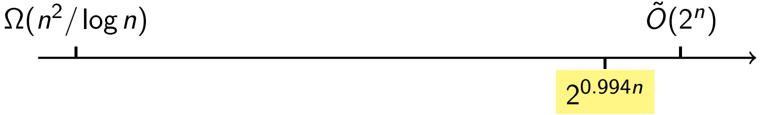
$\Omega(n^2/\log n)$ $\tilde{O}(2^n)$

Linear secret sharing for any monotone function:

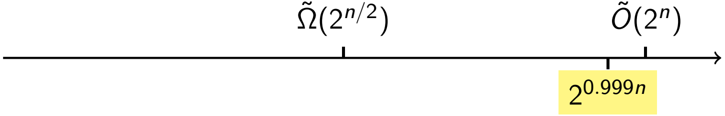
$\tilde{\Omega}(2^{n/2})$ $\tilde{O}(2^n)$

To Summarize

Secret sharing for any monotone function:



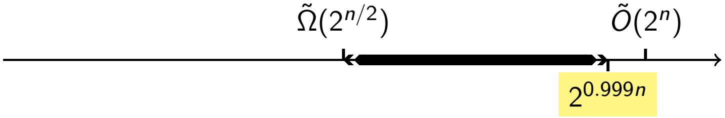
Linear secret sharing for any monotone function:



To Summarize

Secret sharing for any monotone function:

Linear secret sharing for any monotone function:



To Summarize

All Monotone Functions

$\forall F$ has a secret sharing scheme with share size $2^{0.994n}$.

$\forall F$ has a linear secret sharing scheme with share size $2^{0.999n}$.

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All Monotone Functions

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Slice Functions [LVW'18,BKN'18]

Every slice function (there are $2^{\binom{n}{n/2}}$ of them) has a secret sharing scheme with share size $2^{\tilde{O}(\sqrt{n})}$.

To Summarize

All Monotone Functions

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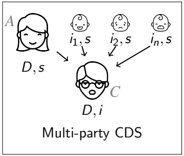

[LVW'18]

To Summarize

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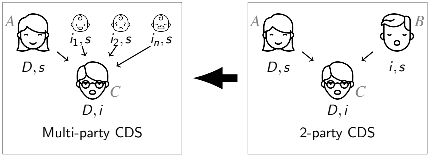

[LVW'18] [LVW'17]

To Summarize

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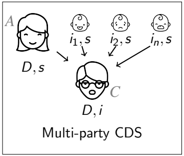
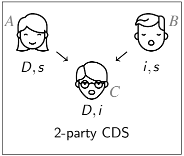
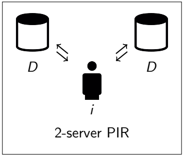


[LVW'18]



[LVW'17]



[Yek'08,Efr'09,DG'15]

To Summarize

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$\forall F$ has a linear secret sharing scheme with share size $2^{0.999n}$.

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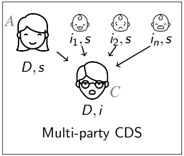

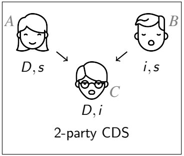

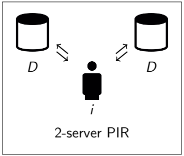

Matching Vectors, OR-poly

[LVW'18] [LVW'17] [Yek'08,Efr'09,DG'15] [BBR'94]