The t-wise Independence of Substitution-Permutation Networks

Tianren Liu[1] Stefano Tessaro[1] Vinod Vaikuntanathan[2]

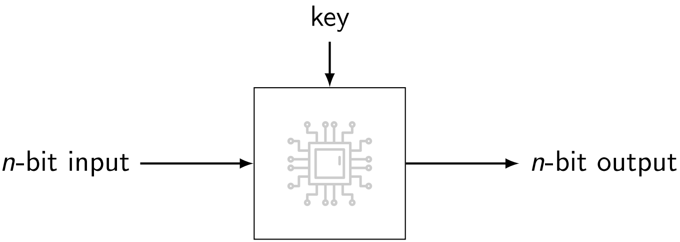
[1]University of Washington, Seattle

[2]MIT, Cambridge

CRYPTO 2021

Random-looking Keyed Permutation

indistinguishable from a random permutation



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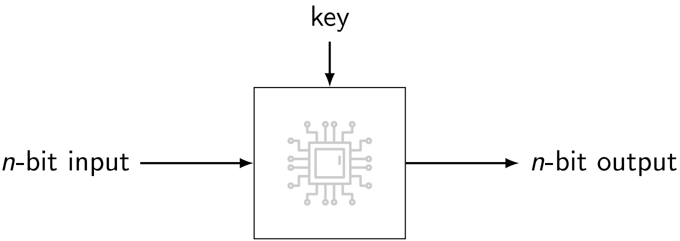

theory

Pseudorandom Permutation

Provable security
based on hardness assumptions

practice

Block Cipher

Heuristic security
resisting known attacks

theory

Pseudorandom Permutation

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Pseudorandom Permutation

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provable security based on . . .

Feistel [LR88] plus
- one-way functions [GGM84]
- factoring [NR04,. . .]
- lattice problems [BPR12,. . .]

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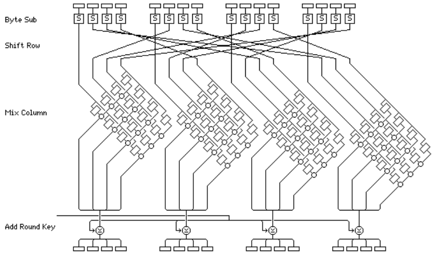
very efficient ciphers (e.g. AES)



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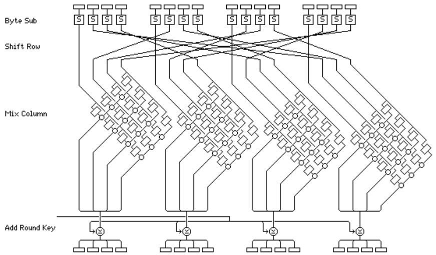

Is AES secure? Is AES secure?

theory

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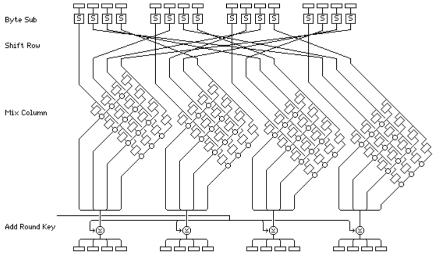

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Base AES on assumptions?

Idealized model

BKL+12, Ste12, ABD+13, LS14, CS14, CLL+14, HT16, DSSL16, GL15, DKS+17, CDK+18, CL18, WYCD20, etc

practice

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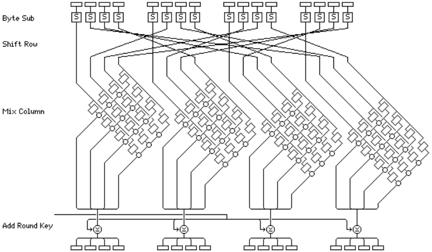

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Cryptanalysis

linear [MY92] and differential [BS91] cryptanalysis, higher-order [Lai94] and truncated [Knu94] differential attacks, impossible differential attacks [Knu98], algebraic attacks [JK97], integral cryptanalysis [KW02], biclique attacks [BKR11], etc

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Provable bounds

on the advantage of known attacks

NK95, KMT01, PSC+02, PSLL03, Kel04, KS07, etc

Prove bounds against an attack class

integral cryptanalysis

algebraic attacks

truncated higher-order differential attacks

biclique attacks

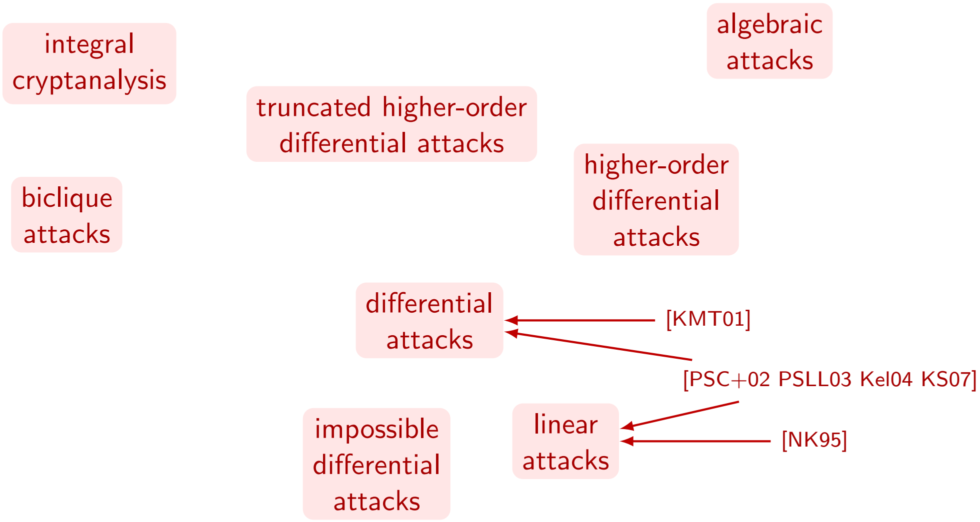
higher-order differential attacks

differential attacks

impossible differential attacks

linear attacks

Prove bounds against an attack class



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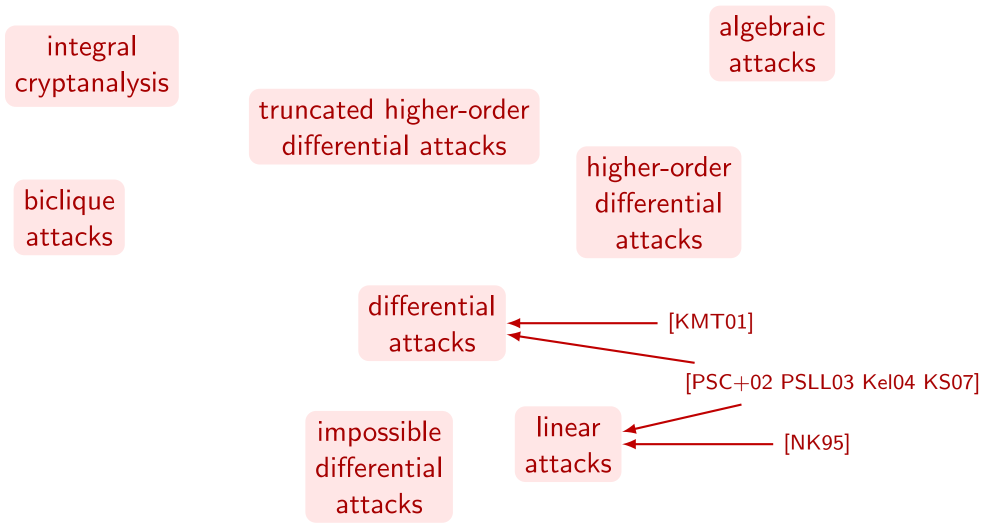

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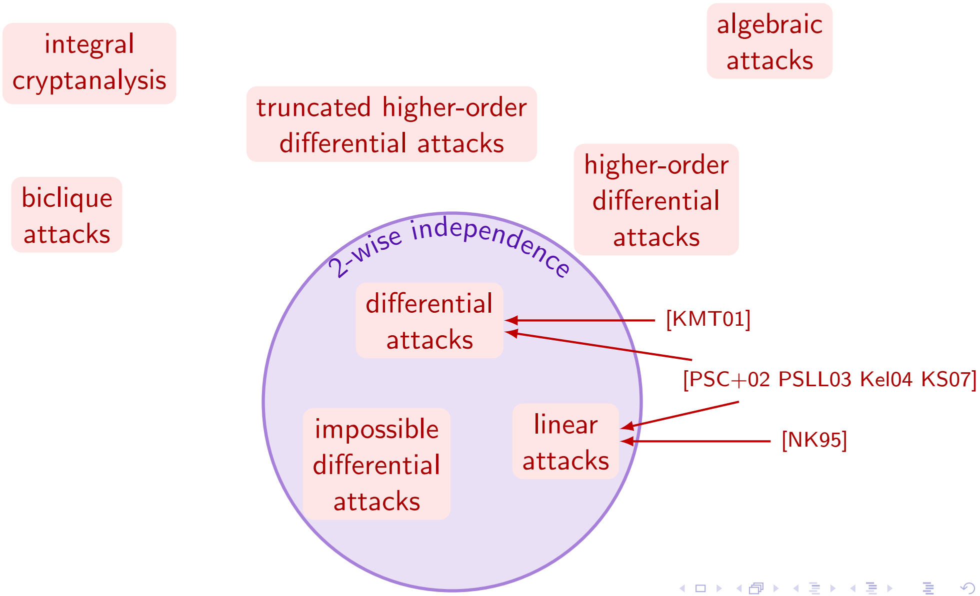

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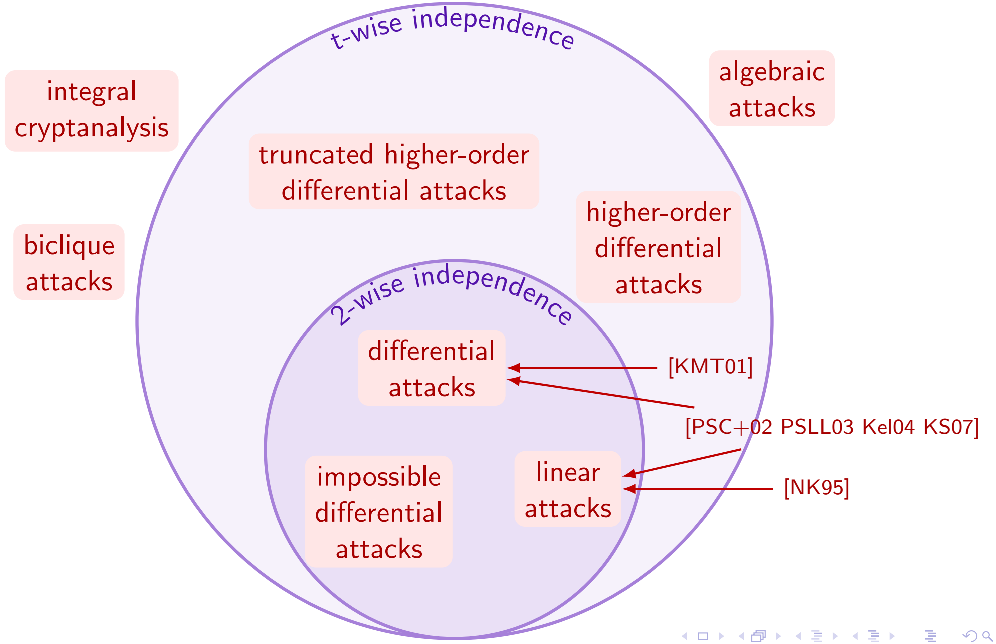
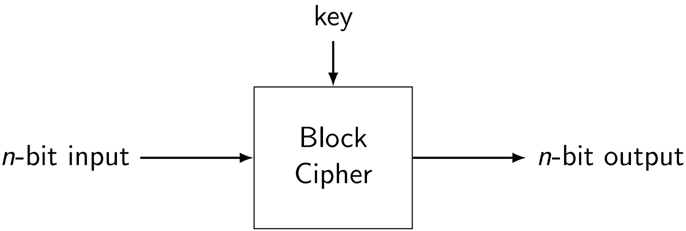


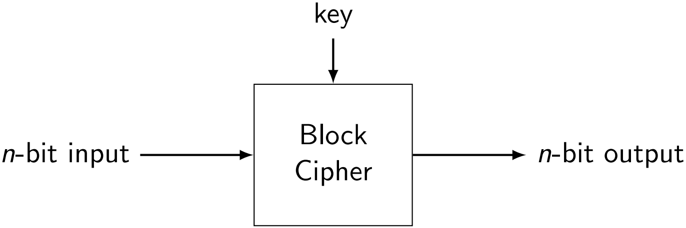


t-wise Independence

$\forall \mathrm{input}_1, \ldots, \mathrm{input}_t$

$\mathrm{output}_1, \ldots, \mathrm{output}_t$ are i.i.d. uniform

used in [HMMR05, KNR05, BH08, AL13]



ε-close to t-wise Independence

$$\forall \text{input}_1, \ldots, \text{input}_t$$
$$\text{StatisticalDistance}((\text{output}_1, \ldots, \text{output}_t), \text{uniform}) \leq \varepsilon$$

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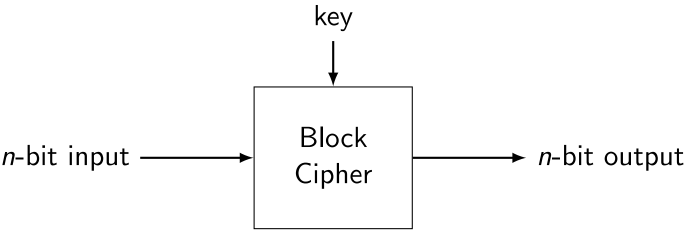

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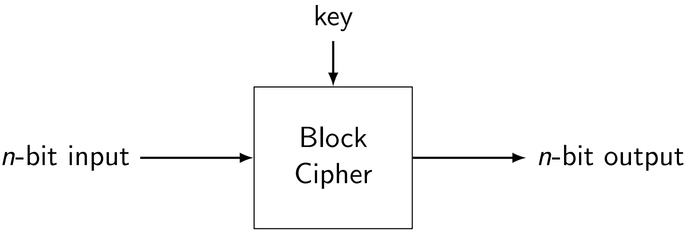

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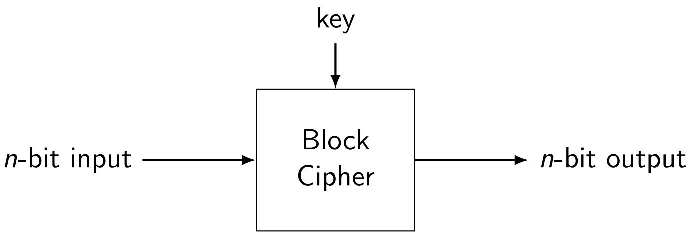

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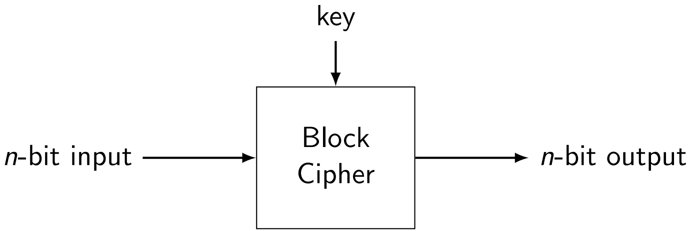

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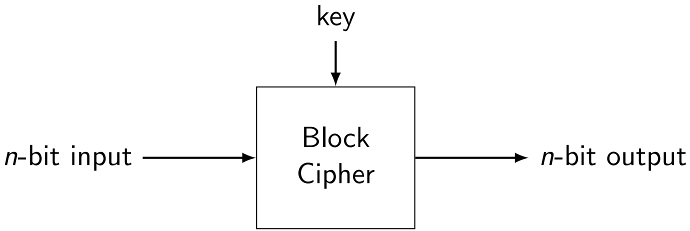

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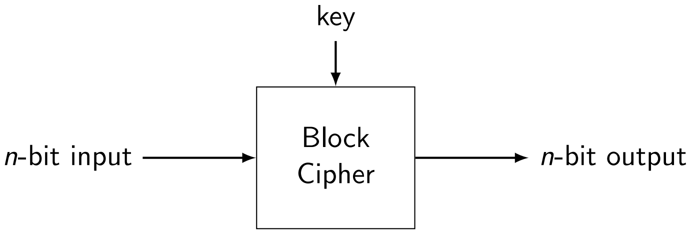
- 2^d non-adaptive queries **order-d differential attacks**



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Statistically indistinguishable with t non-adaptive queries

$$\varepsilon\text{-close to 2-wise indp} \implies \begin{cases} \text{MEDP} \leq \varepsilon + \frac{1}{2^n - 1} & \text{(differential attack)} \\ \text{CORR} \leq 8\varepsilon + \frac{4}{2^n} & \text{(linear attack)} \end{cases}$$

Key-Alternating Cipher (KAC)

Substitution-Permutation Network (SPN)

Advanced Encryption Standard (AES)

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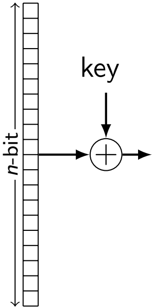

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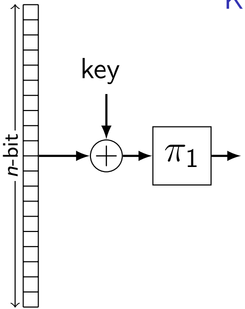

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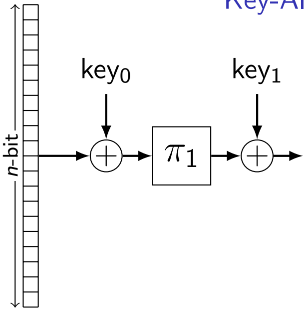

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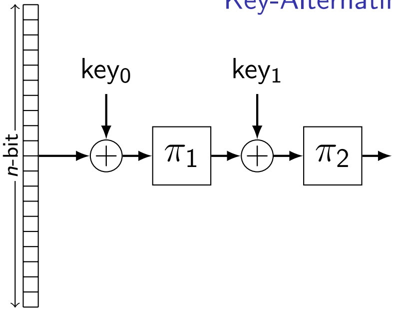

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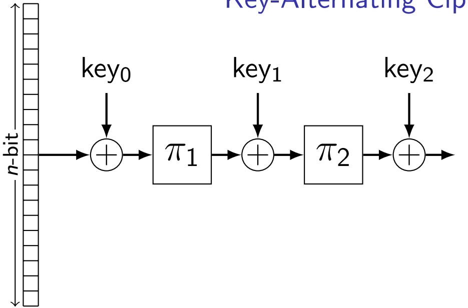

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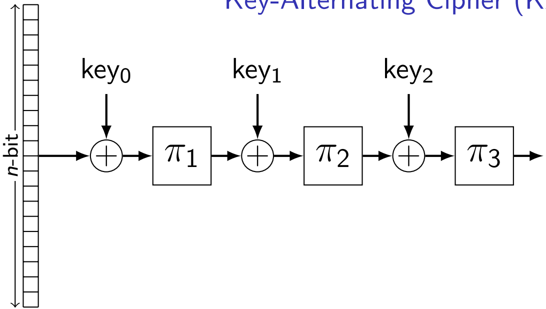

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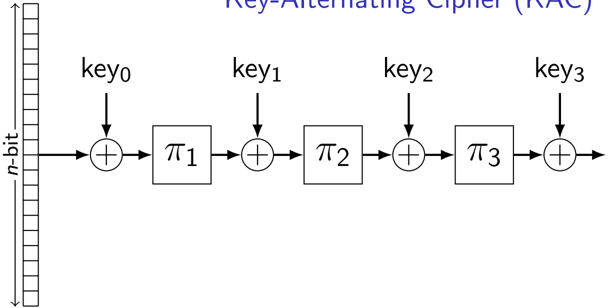

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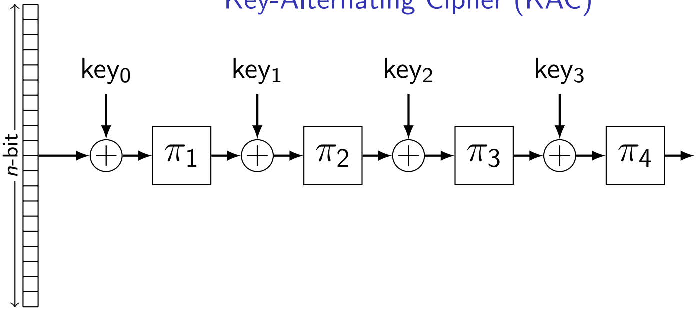

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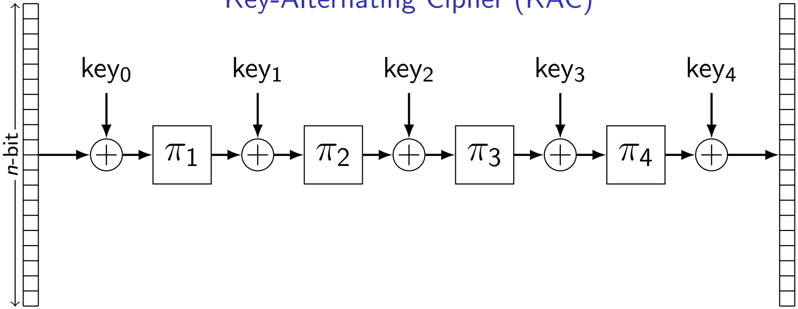

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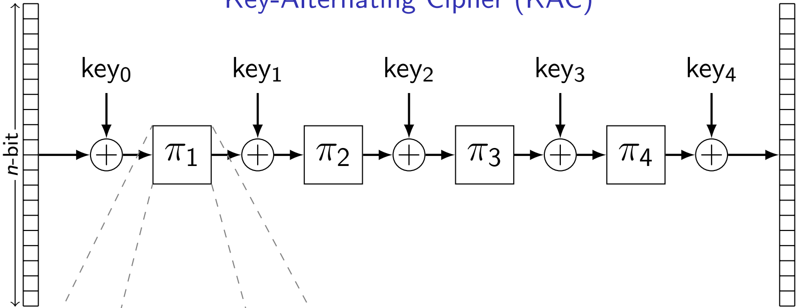

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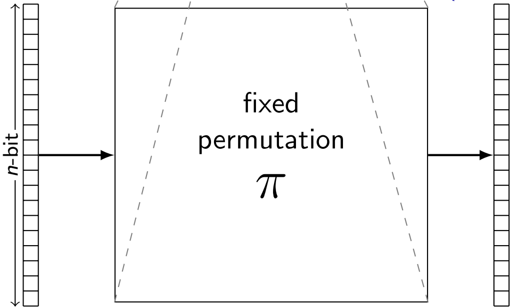

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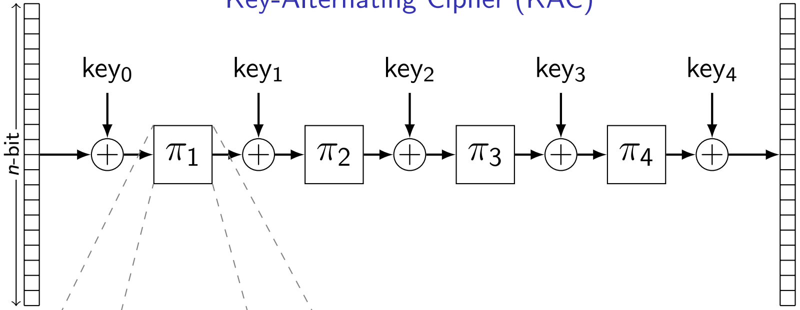

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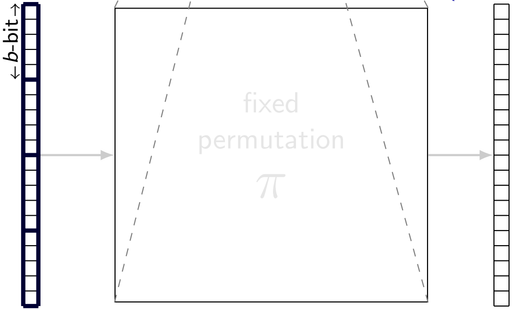

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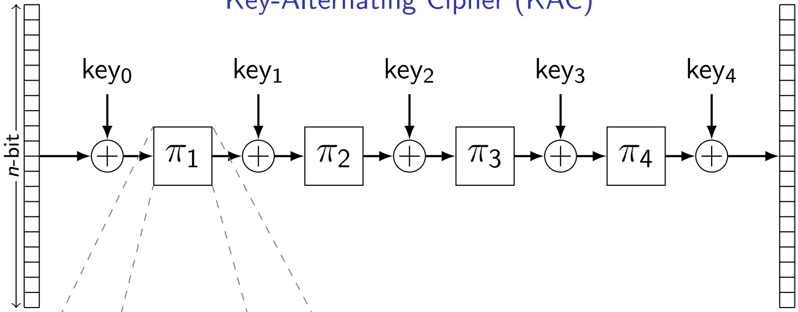

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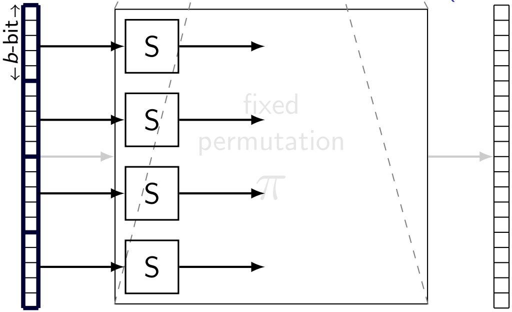

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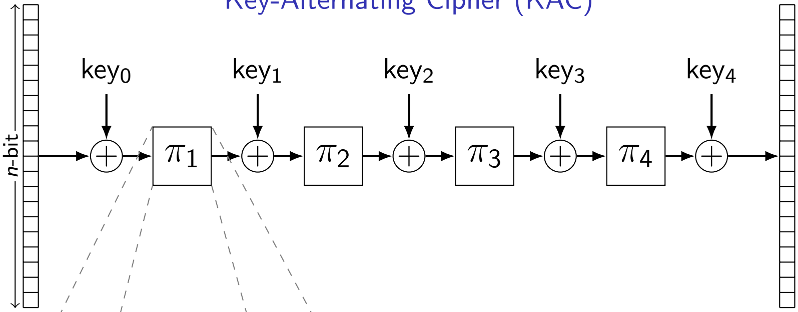

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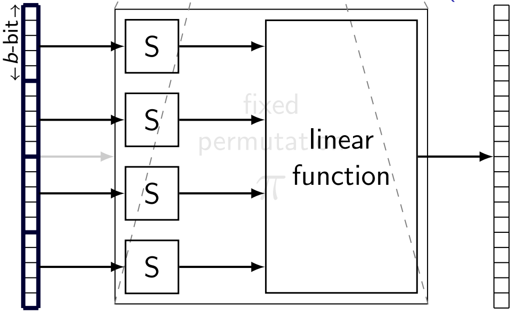

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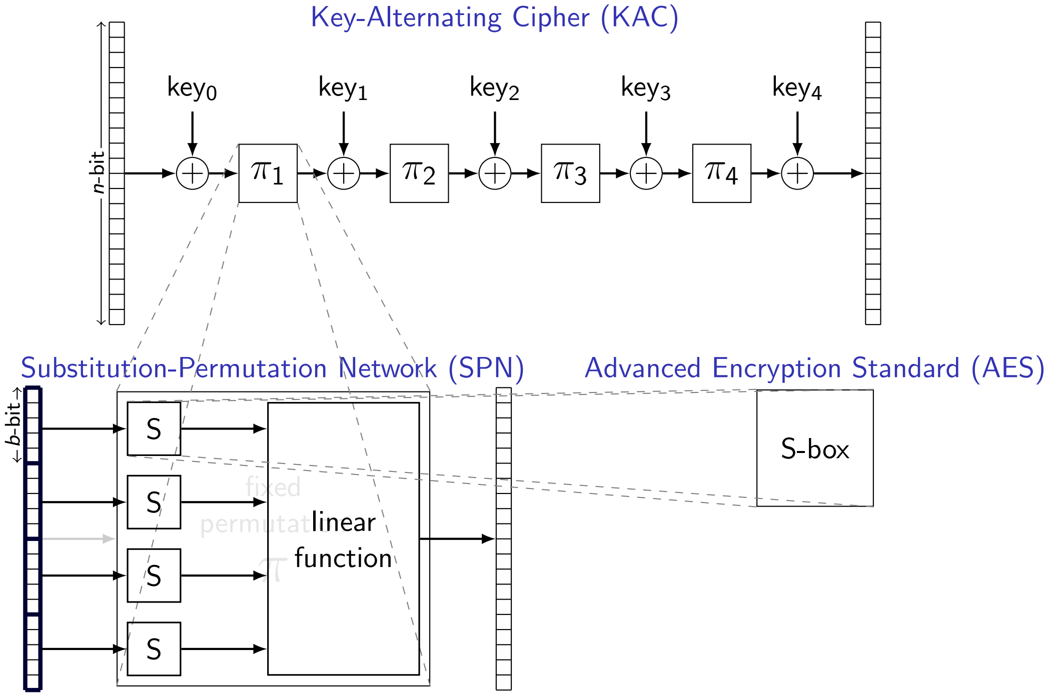
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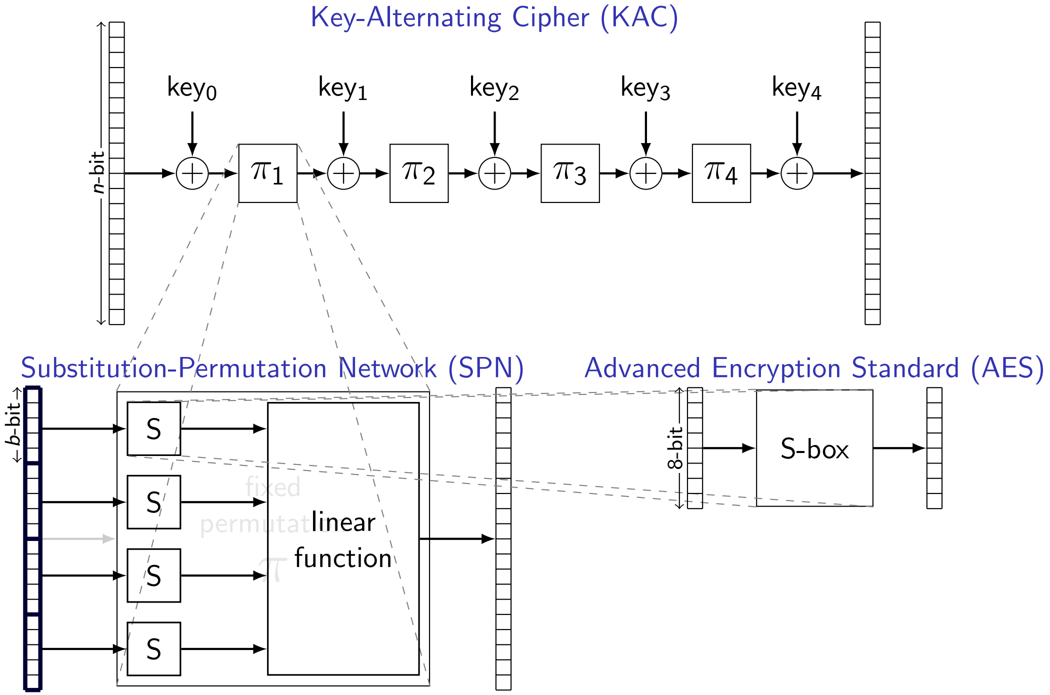
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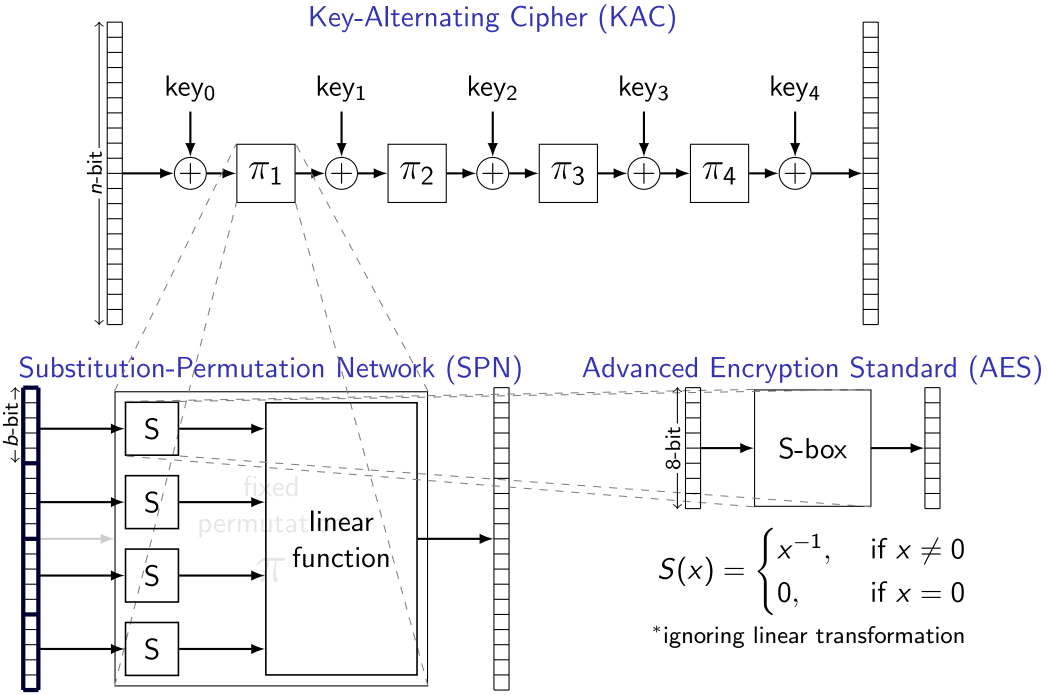

$$S(x) = \begin{cases} x^{-1}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

*ignoring linear transformation

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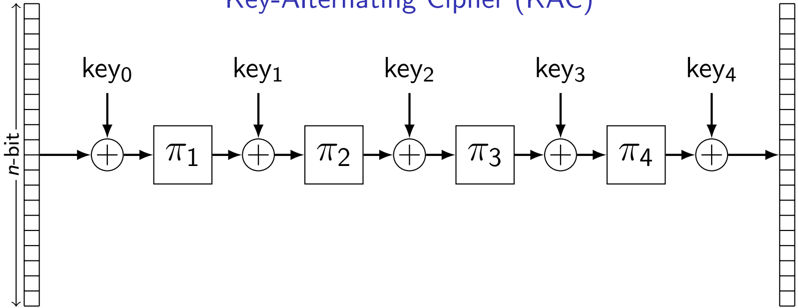

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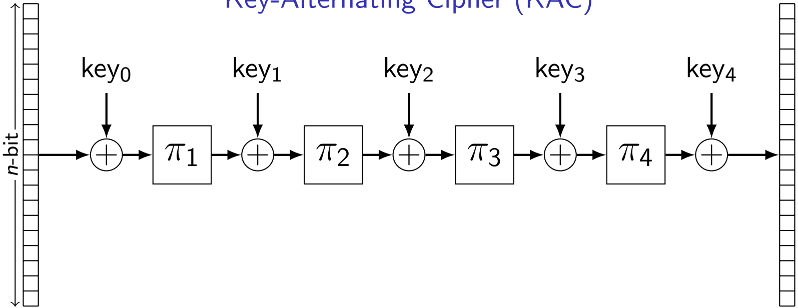
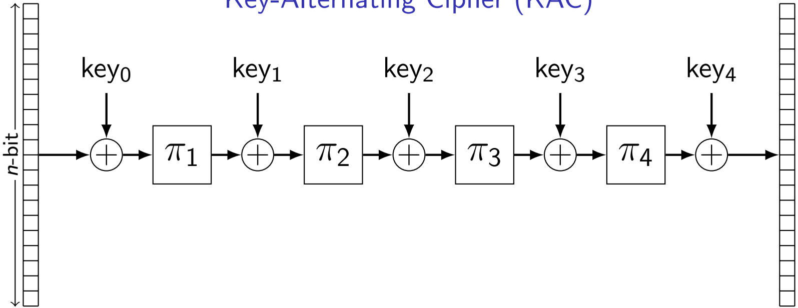


r-round $\mathrm{KAC}(\pi_1, \ldots, \pi_r)$ is not $(r+2)$-wise independent

Key-Alternating Cipher (KAC)



Our Results (KAC)

r-round KAC$(\pi_1, \ldots, \pi_r)$ is close to $(r - o(r))$-wise independent for most $\pi_1, \ldots, \pi_r$

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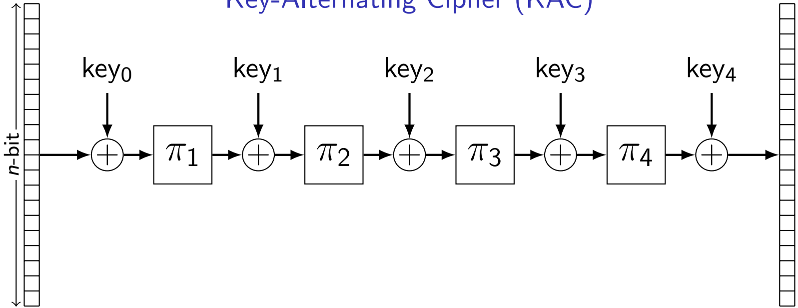

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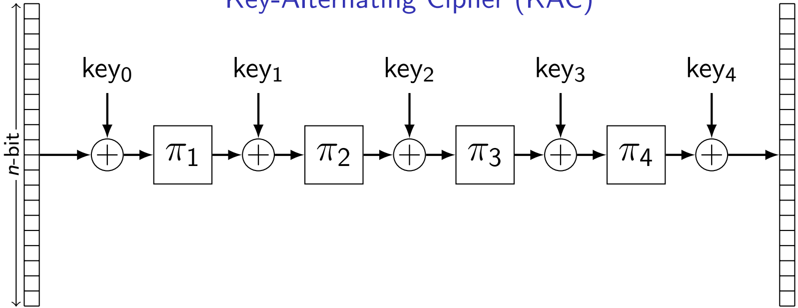

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*existential result & probabilistic method
*unlike ideal model results, $\pi_1, \ldots, \pi_r$ are completely known to adv

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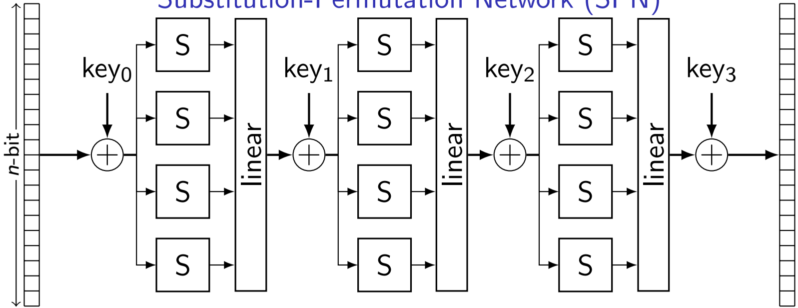

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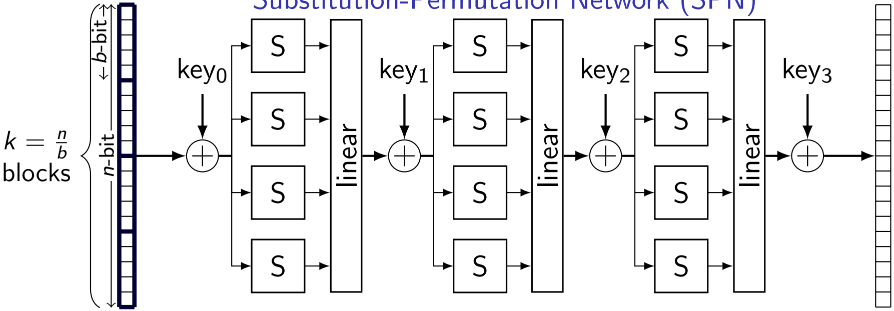

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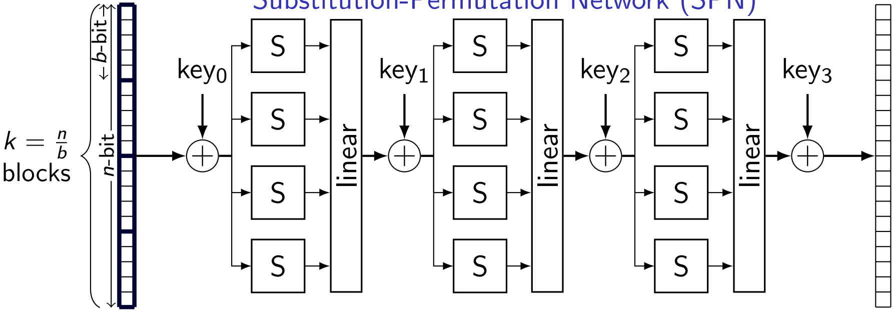
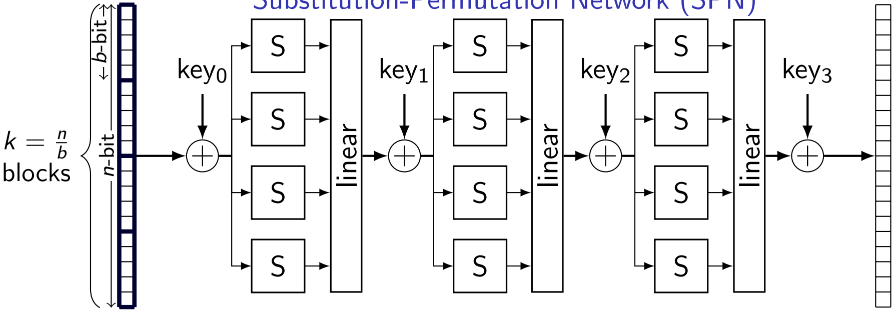


$S(x) = x^{-1}$ (used by AES) *or* $S(x) = x^3$ (used by MiMC)

Substitution-Permutation Network (SPN)



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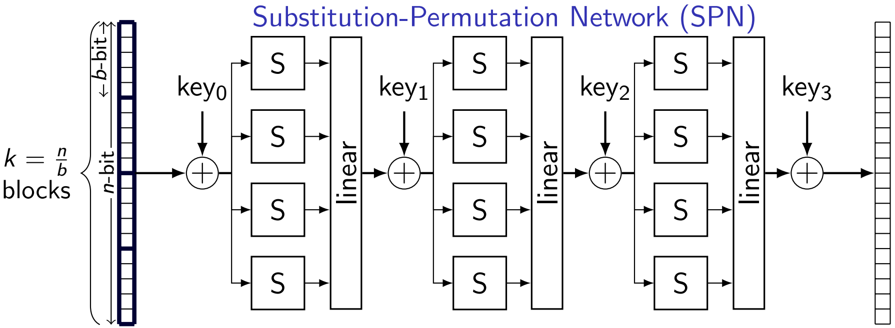
Our Results

2-round SPN is $(\frac{4k}{2^b} + \sqrt{\frac{2^k}{2^b}})$-close to 2-wise independent.

3-round SPN is $(\frac{8k}{2^b} + \sqrt{\frac{k}{2^b}})$-close to 2-wise independent.



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State of the art [Park-Sung-Lee-Lim 03]

4-round AES is pointwise 2^{17}-close to 2-wise independent.

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def pointwise ε-close to uniform

$$1 - \varepsilon \leq \frac{\Pr[X \leftarrow \text{distribution}; X = v]}{\Pr[X \leftarrow \text{uniform}; X = v]} \leq 1 + \varepsilon$$

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MPR Amplification Lemma [Maurer-Pietrzak-Renner 07]

$$\left.\begin{array}{l}\mathcal{F} \text{ is } \varepsilon\text{-close to 2-wise indp.} \\ \mathcal{G} \text{ is } \delta\text{-close to 2-wise indp.}\end{array}\right\} \implies \mathcal{F} \circ \mathcal{G} \text{ is } 2\varepsilon\delta\text{-close to 2-wise indp.}$$

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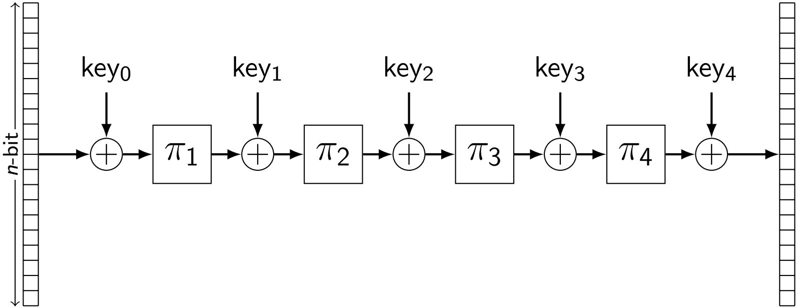
Amplifying Our Results

$6r$-round AES is $(2^{r-1}0.472^r)$-close to 2-wise independent.

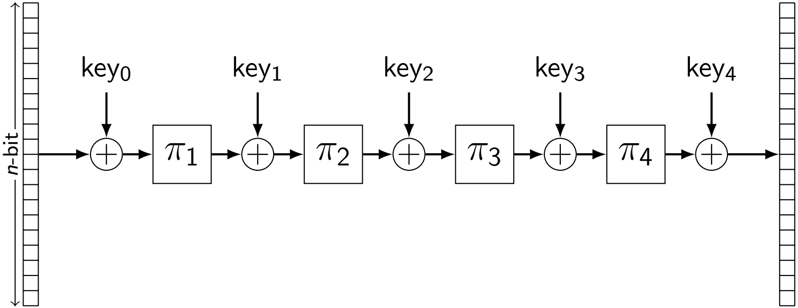
Proof Overview (KAC)



Our Results (KAC)

r-round KAC$(\pi_1, \ldots, \pi_r)$ is close to
$(r - o(r))$-wise independent
for most $\pi_1, \ldots, \pi_r$

Proof Overview (KAC)



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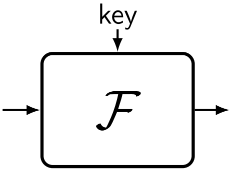
Prove by induction

Proof Overview (KAC)

$\mathcal{F}$ is t-wise indp.



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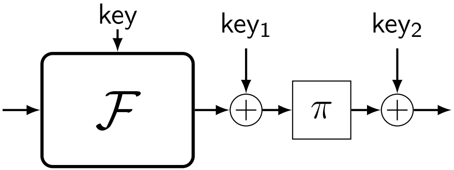

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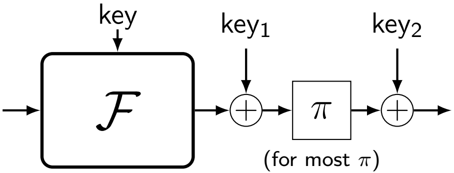

is pointwise $O(t^2 n)$-close to $(t+1)$-wise indp.

Proof Overview (KAC)

Independence Amplification Lemma

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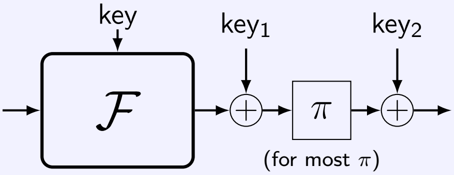

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*existential result & probabilistic method on π

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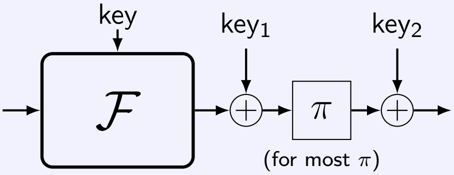

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$\forall \text{input}_1, \ldots, \text{input}_t, \text{output}_1, \ldots, \text{output}_t$

$$\frac{1-\varepsilon}{2^{tn}} \leq \Pr[\text{output}_1, \ldots, \text{output}_t] \leq \frac{1+\varepsilon}{2^{tn}}$$

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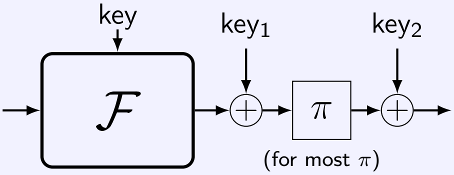

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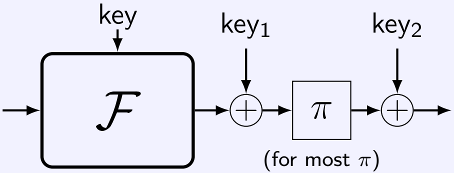

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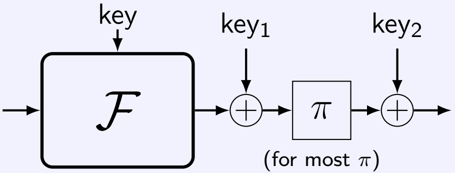

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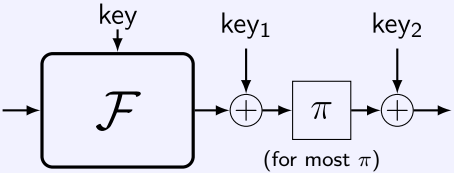

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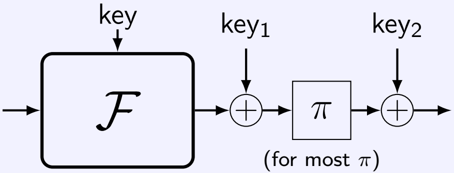

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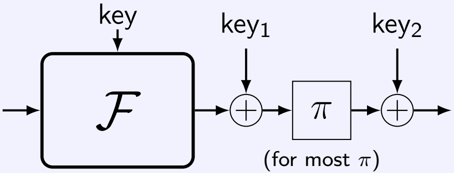

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$\Downarrow$

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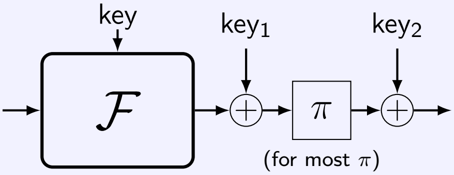
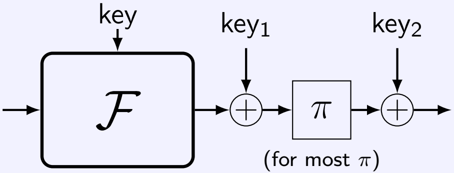


is pointwise $O((1+\varepsilon)t^2 n)$-close to $(t+1)$-wise indp.

Distance Amplification Lemma

$\mathcal{F}$ is pointwise **very** close to t-wise indp. & pointwise **somewhat** close to $(t+1)$-wise indp. $\Longrightarrow$

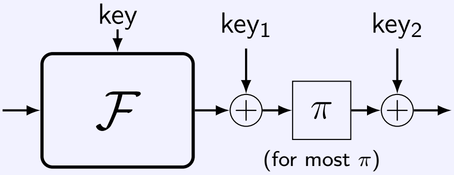


is pointwise **very** close to $(t+1)$-wise indp.

Proof Overview (KAC)

Independence Amplification Lemma

$\mathcal{F}$ is pointwise ε-close to t-wise indp. $\implies$



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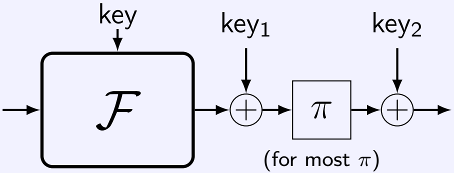

is pointwise $(\varepsilon + \frac{O(\varepsilon' t)}{2^{n/3}})$-close to $(t+1)$-wise indp.

Proof Overview (KAC)

number of rounds	0-round	1-round	2-round	3-round	4-round
closeness to 1-wise indp.					
closeness to 2-wise indp.					
closeness to 3-wise indp.					
closeness to 4-wise indp.					
closeness to 5-wise indp.					

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number of rounds	0-round	1-round	2-round	3-round	4-round
closeness to 1-wise indp.	0	0	0	0	0
closeness to 2-wise indp.					
closeness to 3-wise indp.					
closeness to 4-wise indp.					
closeness to 5-wise indp.					

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closeness to 2-wise indp.		$O(n)$			
closeness to 3-wise indp.			$O(n^2)$		
closeness to 4-wise indp.				$O(n^3)$	
closeness to 5-wise indp.					$O(n^4)$

Independence Amplification

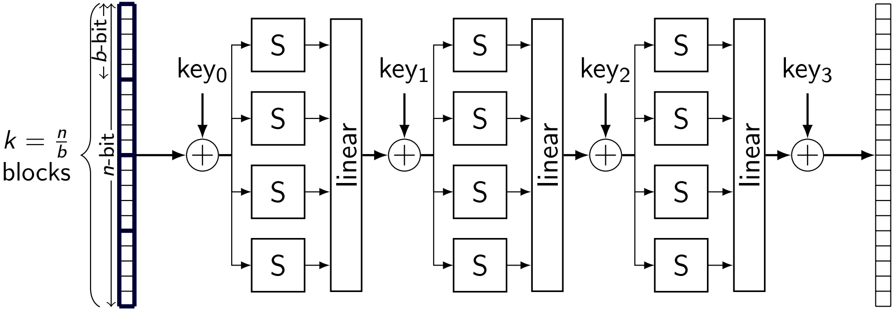
Proof Overview (KAC)

number of rounds	0-round	1-round	2-round	3-round	4-round
closeness to 1-wise indp.	0	0	0	0	0
closeness to 2-wise indp.		$O(n)$	$O(\frac{n}{2^{n/3}})$	$O(\frac{n}{2^{2n/3}})$	$O(\frac{n}{2^n})$
closeness to 3-wise indp.			$O(n^2)$	$O(\frac{n^2}{2^{n/3}})$	$O(\frac{n^2}{2^{2n/3}})$
closeness to 4-wise indp.				$O(n^3)$	$O(\frac{n^3}{2^{n/3}})$
closeness to 5-wise indp.					$O(n^4)$

Independence Amplification

Distance Amplification

Proof Overview (SPN & AES)



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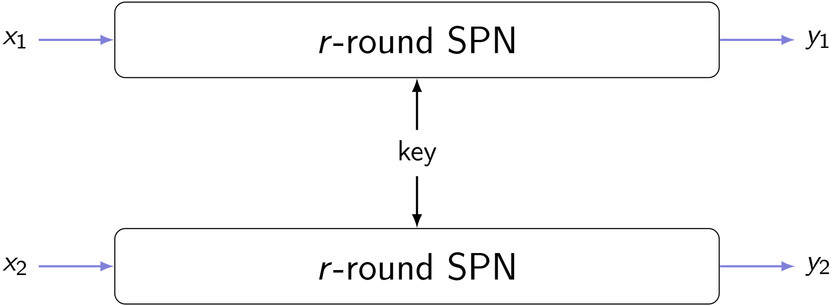

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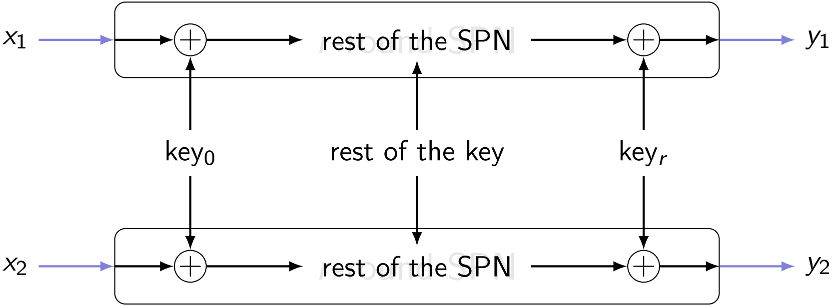

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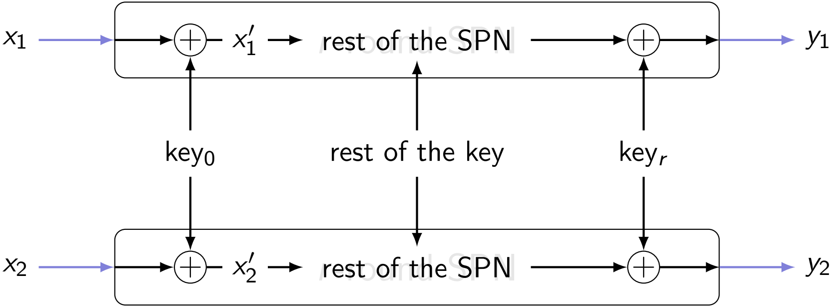

(x_1', x_2') is random conditioning on $x_1' - x_2' = x_1 - x_2$

Proof Overview (SPN & AES)

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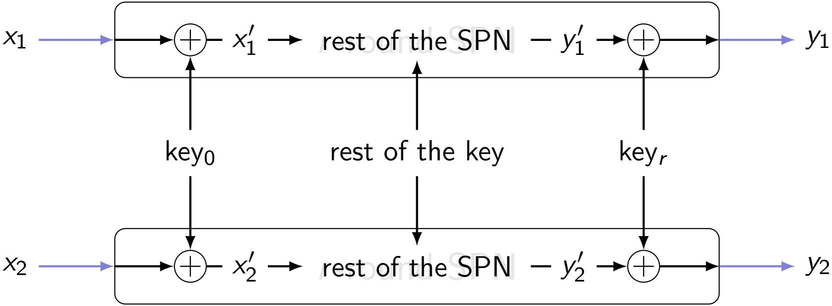
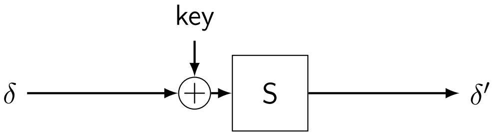


(x'_1, x'_2) is random conditioning on $x'_1 - x'_2 = x_1 - x_2$

$$\mathrm{SD}((y_1, y_2), \mathrm{uniform}) = \mathrm{SD}(y'_1 - y'_2, \mathrm{uniform})$$

Proof Overview (SPN & AES)

S-box: input difference $\delta \quad \mapsto \quad$ output difference δ'



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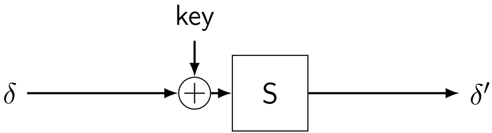

given inputs x_1, x_2 s.t. $x_1 \oplus x_2 = \delta$,
what is the distribution of $\delta' = S(x_1 \oplus \text{key}) \oplus S(x_2 \oplus \text{key})$?

Proof Overview (SPN & AES)

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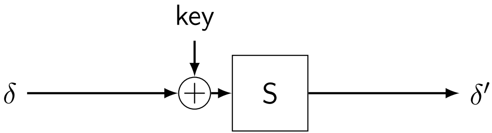

$$S(x) = x^{-1} \quad or \quad S(x) = x^3 \quad \text{over } \mathbb{F}_{2^b}$$

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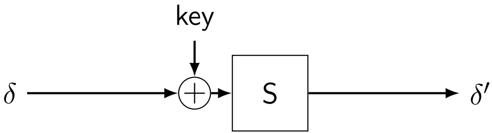

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Subspace Sampling Lemma

View δ, δ' as dimension-n vectors in $\mathbb{F}_2^b$

δ' is a random vector orthogonal to δ!

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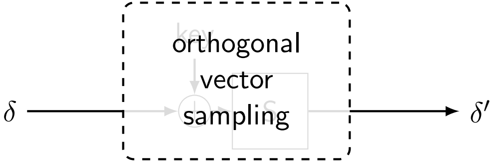

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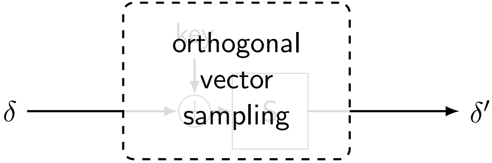

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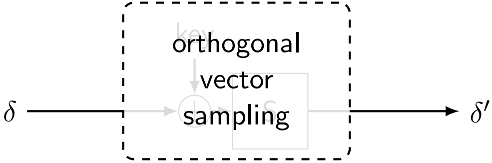

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- $\delta = 0 \implies \delta' = 0$

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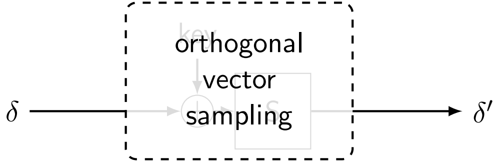

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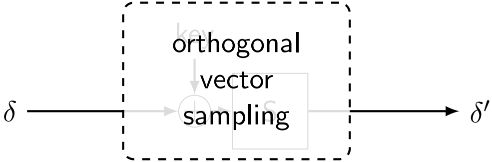
- $\delta = 0 \implies \delta' = 0$
- $\exists \pi, \pi'$ s.t. $\pi(\delta')$ is a random vector orthogonal to $\pi'(\delta)$

Proof Overview (SPN & AES)

S-box: input difference δ $\mapsto$ output difference δ'



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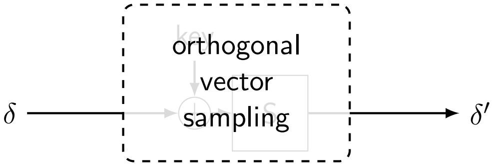
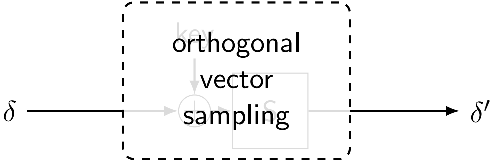
Not really true. Actually ...

- $\delta = 0 \implies \delta' = 0$
- $\exists \pi, \pi'$ s.t. $\pi(\delta')$ is a random vector orthogonal to $\pi'(\delta)$

Proof Overview (SPN & AES)

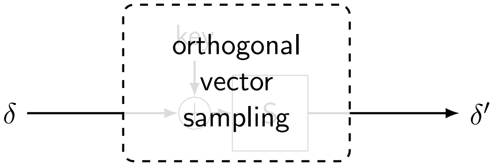


Proof Overview (SPN & AES)



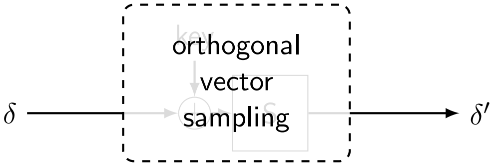
fixed $\delta \neq 0$ $\implies$ $H_\infty(\delta') = b - 1$

Proof Overview (SPN & AES)



$$H_\infty(\delta) \geq b - 1 \qquad \Longrightarrow \qquad ???$$

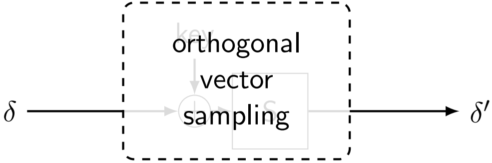
Proof Overview (SPN & AES)



Extraction Lemma

$H_\infty(\delta) \geq b - 1 \quad \Longrightarrow \quad \delta'$ close to uniform

Proof Overview (SPN & AES)

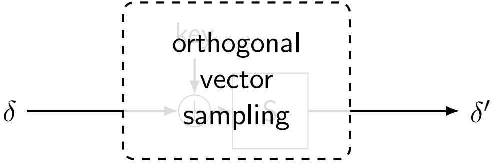


Extraction Lemma

$$\mathsf{H}_\infty(\delta) \geq b - 1 \quad \Longrightarrow \quad \delta' \text{ close to uniform}$$

Proved by Fourier analysis

Proof Overview (SPN & AES)



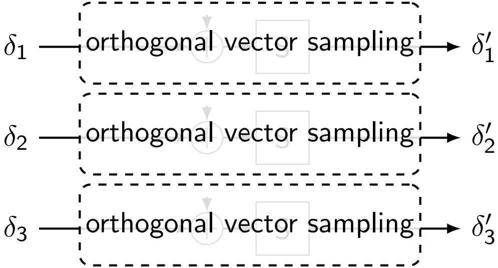
Extraction Lemma

$\mathsf{H}_\infty(\delta) \geq b - 1 \quad \Longrightarrow \quad \delta'$ close to uniform

Proved by Fourier analysis

(full version) Proved by collision probability

Proof Overview (SPN & AES)



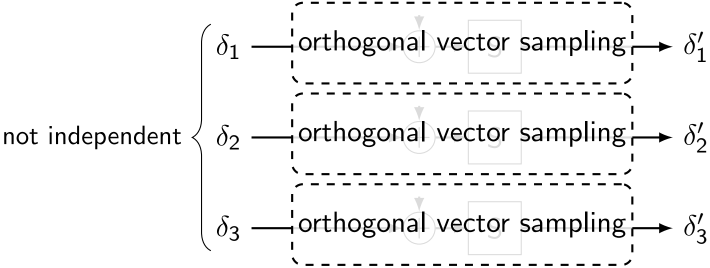
Extraction Lemma

$\forall i\ \mathrm{H}_\infty(\delta_i) \geq b - 1 \quad \Longrightarrow \quad (\delta'_1, \ldots, \delta'_k)$ close to uniform

Proved by Fourier analysis

(full version) Proved by collision probability

Proof Overview (SPN & AES)



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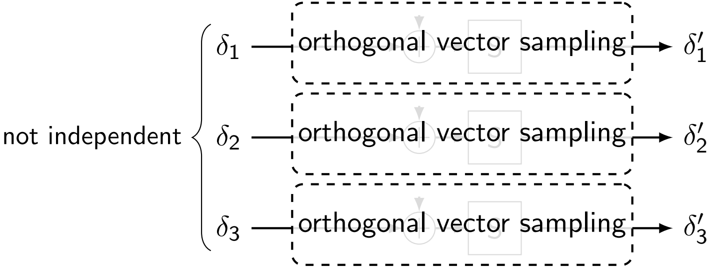


Extraction Lemma

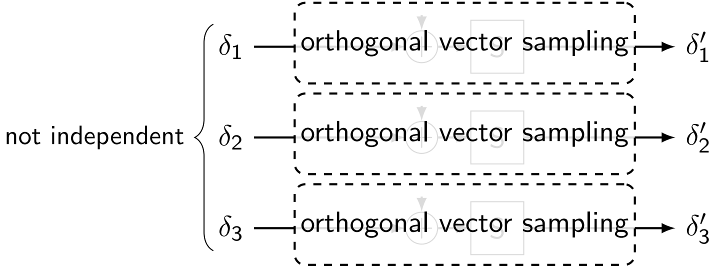
$\forall i\ \mathrm{H}_\infty(\delta_i) \geq b - 1 \implies (\delta'_1, \ldots, \delta'_k)$ close to uniform

$\mathrm{H}_\infty(\{\delta_i\}_{i \in S}) \geq (b-1) \cdot |S|$ for any subset $S \subseteq [k]$ $\implies (\delta'_1, \ldots, \delta'_k)$ very close to uniform

Proved by Fourier analysis

(full version) Proved by collision probability

Proof Overview (SPN & AES)



Extraction Lemma

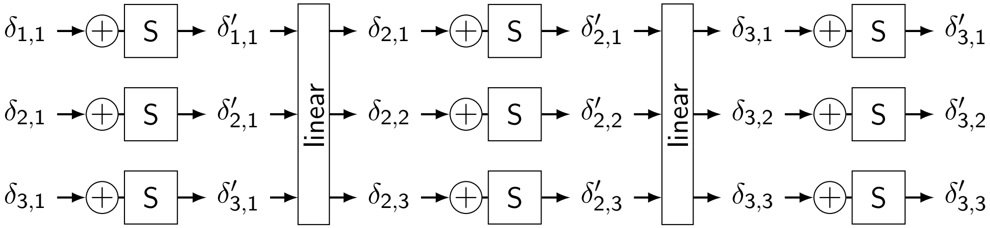
$$\forall i \; H_\infty(\delta_i) \geq b - 1 \implies \mathrm{SD}((\delta'_1, \ldots, \delta'_k), \text{uniform}) \leq \sqrt{\frac{2^k - 1}{2^b}}$$

$$H_\infty(\{\delta_i\}_{i \in S}) \geq (b-1) \cdot |S| \text{ for any subset } S \subseteq [k] \implies \mathrm{SD}((\delta'_1, \ldots, \delta'_k), \text{uniform}) \leq \sqrt{\frac{k}{2^b}}$$

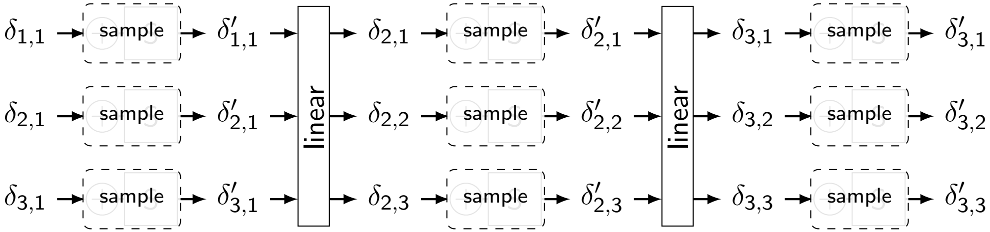
Proved by Fourier analysis

(full version) Proved by collision probability

Proof Overview (SPN & AES)



Proof Overview (SPN & AES)



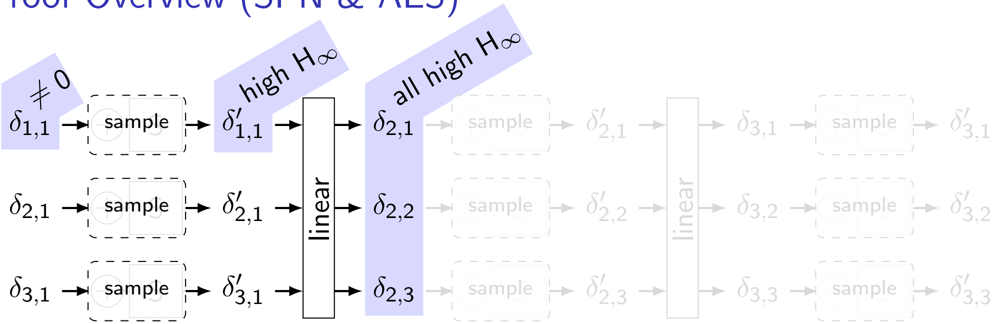
Proof Overview (SPN & AES)

$\xrightarrow{\text{w.l.o.g.}}$ $\delta_{1,1} \neq 0$

Proof Overview (SPN & AES)

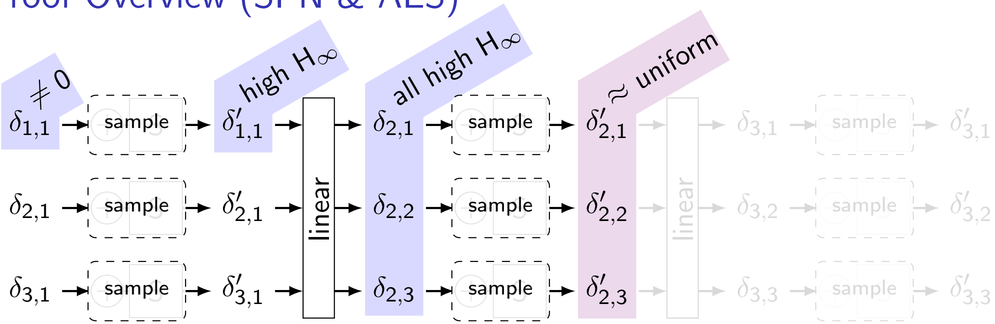
$\overset{\text{w.l.o.g.}}{\Longrightarrow}$ $\delta_{1,1} \neq 0 \Longrightarrow H_\infty(\delta'_{1,1}) = b - 1$

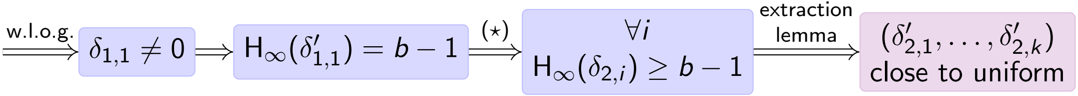
Proof Overview (SPN & AES)



$\xRightarrow{\text{w.l.o.g.}}$ $\delta_{1,1} \neq 0$ $\Longrightarrow$ $H_\infty(\delta'_{1,1}) = b - 1$ $\xRightarrow{(\star)}$ $\forall i$ $H_\infty(\delta_{2,i}) \geq b - 1$

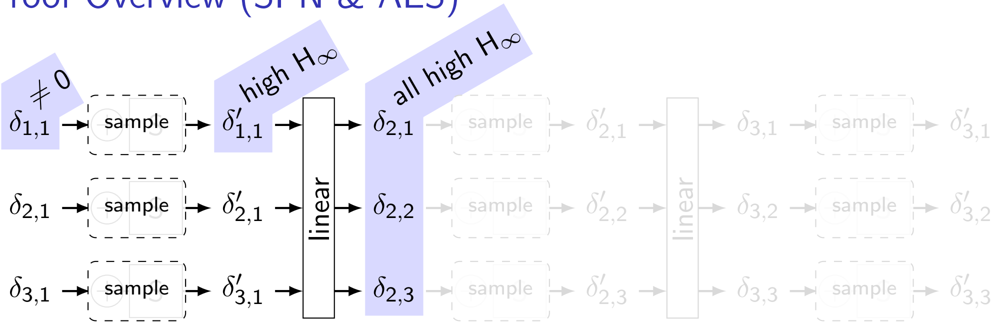
Proof Overview (SPN & AES)





$$SD = \sqrt{\frac{2^k}{2^b}} + O\left(\frac{k}{2^b}\right)$$

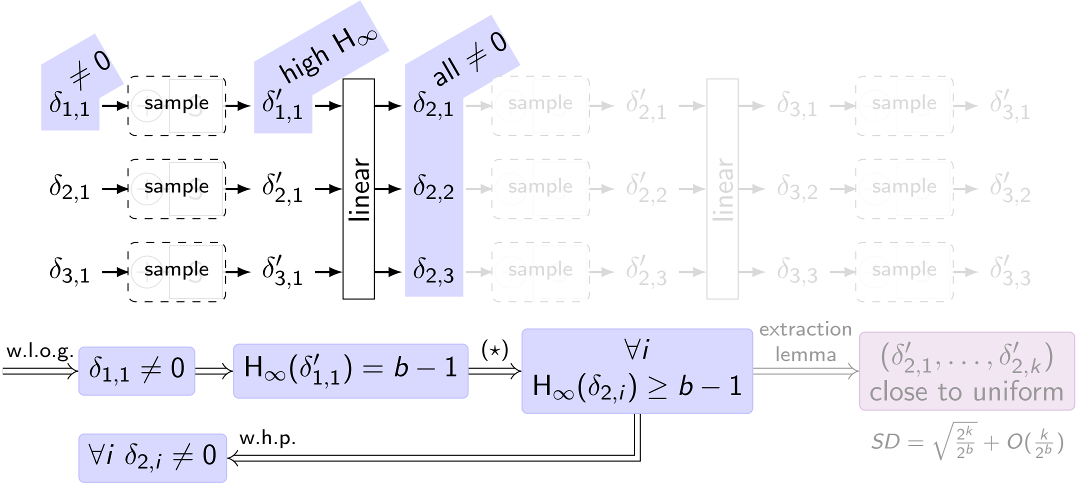
Proof Overview (SPN & AES)



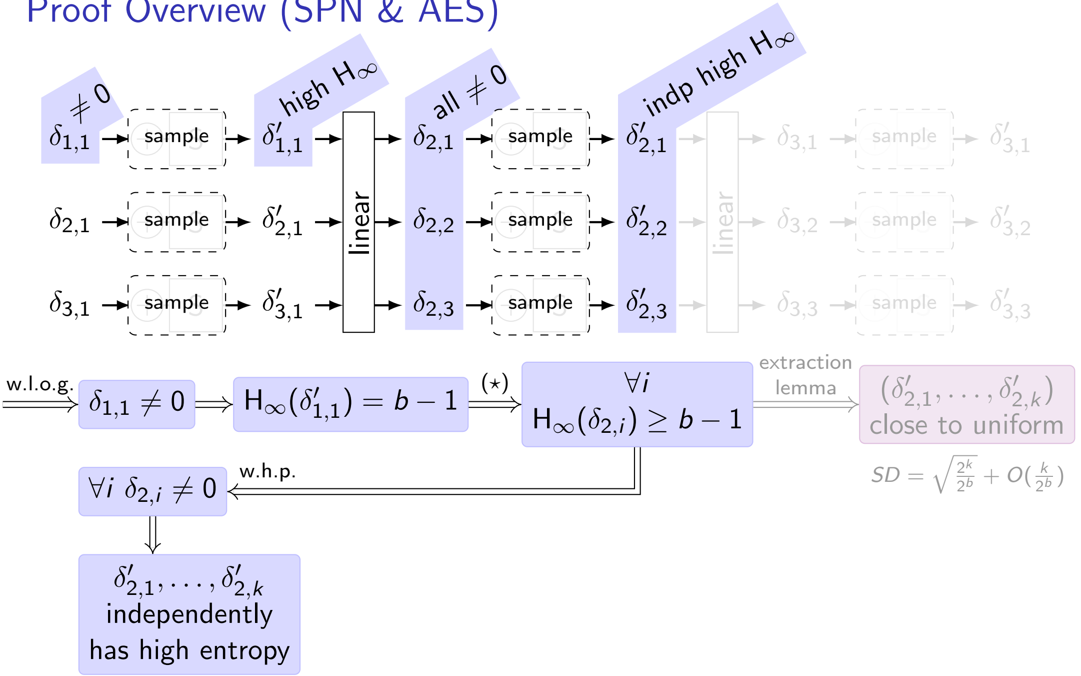
w.l.o.g. $\Longrightarrow$ $\delta_{1,1} \neq 0$ $\Longrightarrow$ $H_\infty(\delta'_{1,1}) = b-1$ $\overset{(\star)}{\Longrightarrow}$ $\forall i$ $H_\infty(\delta_{2,i}) \geq b-1$ $\overset{\text{extraction lemma}}{\Longrightarrow}$ $(\delta'_{2,1}, \ldots, \delta'_{2,k})$ close to uniform

$SD = \sqrt{\frac{2^k}{2^b}} + O(\frac{k}{2^b})$

Proof Overview (SPN & AES)



Proof Overview (SPN & AES)



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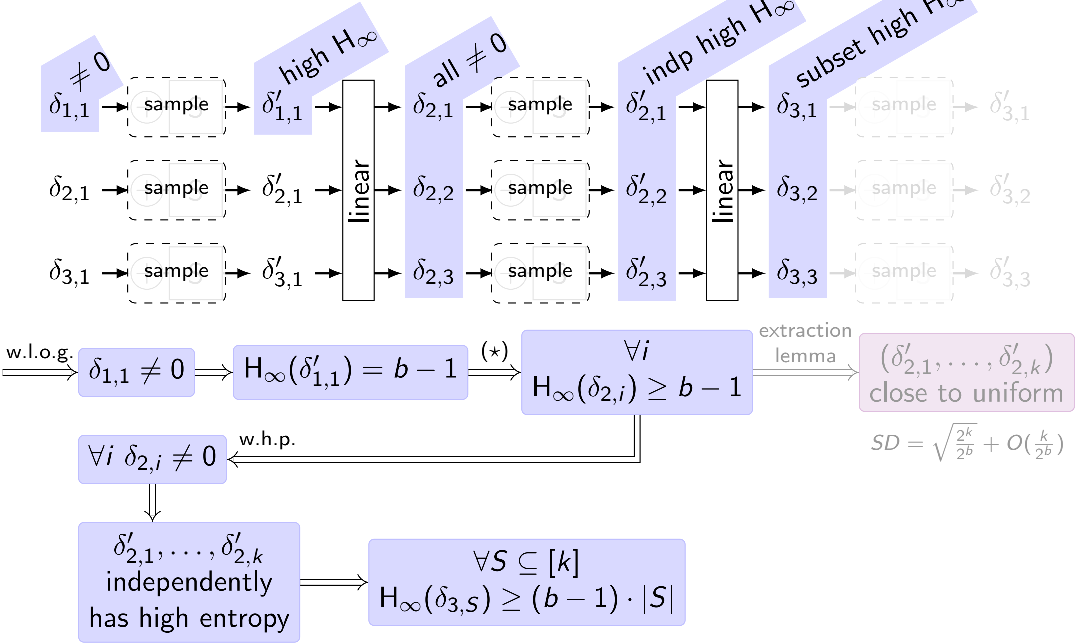
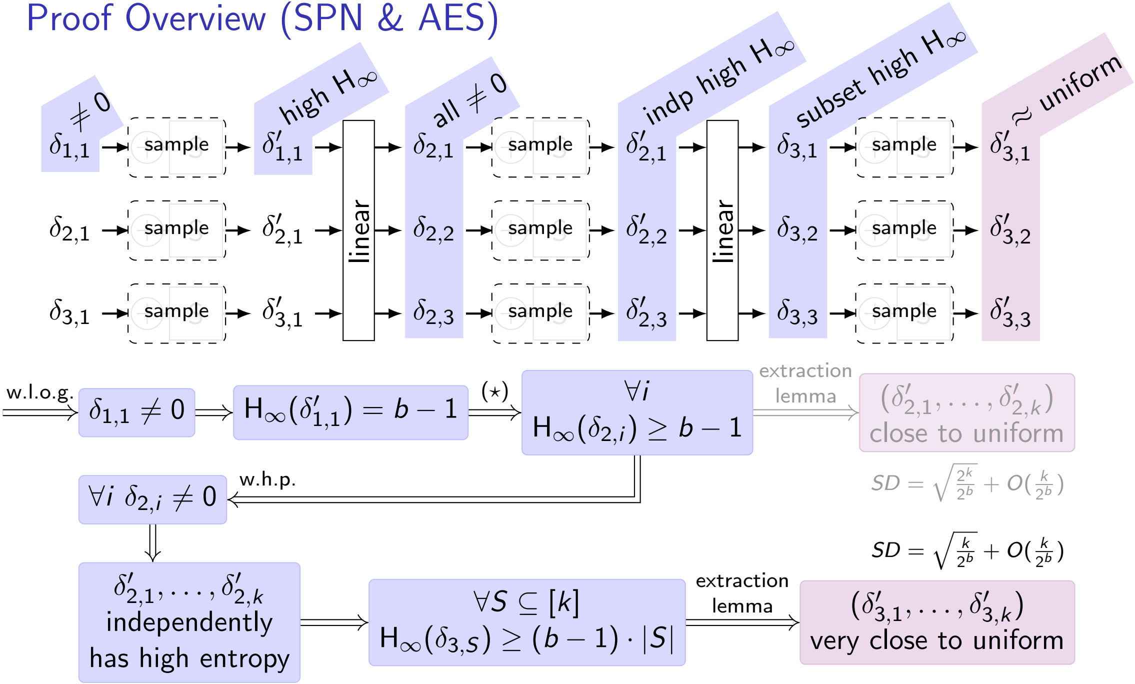


Proof Overview (SPN & AES)



Our Results (SPN & AES)

2-round SPN is $(\frac{4k}{2^b} + \sqrt{\frac{2^k}{2^b}})$-close to 2-wise independent.

3-round SPN is $(\frac{8k}{2^b} + \sqrt{\frac{k}{2^b}})$-close to 2-wise independent.

Our Results (SPN & AES)

2-round SPN is $(\frac{4k}{2^b} + \sqrt{\frac{2^k}{2^b}})$-close to 2-wise independent.

3-round SPN is $(\frac{8k}{2^b} + \sqrt{\frac{k}{2^b}})$-close to 2-wise independent.

6-round AES is 0.472-close to 2-wise independent.

Our Results (KAC)

r-round KAC$(\pi_1, \ldots, \pi_r)$ is close to $(r - o(r))$-wise indp for most $\pi_1, \ldots, \pi_r$

Our Results (SPN & AES)

2-round SPN is $(\frac{4k}{2^b} + \sqrt{\frac{2^k}{2^b}})$-close to 2-wise independent.

3-round SPN is $(\frac{8k}{2^b} + \sqrt{\frac{k}{2^b}})$-close to 2-wise independent.

6-round AES is 0.472-close to 2-wise independent.

t-**wise independence** has a really rich body of problems . . .

- Amplify independence like what we did in KAC
  - 3-wise independence of a concrete cipher
- The role of key scheduling
- Analysis of other concrete cipher design
  - e.g. add–rotate–xor (ARX) cipher
- The relationship between t-wise independent and other class(es) of attack